MA160/190 Algebra Exercises : Semester II

1. (a) Calculate 17^{-1} mod 26.

   (b) Calculate A^{-1} mod 26 where $A = \begin{pmatrix} 15 & 3 \\ 19 & 24 \end{pmatrix}$. Express each of the four entries of A^{-1} as an integer in the range 0 to 25.

   (c) The ciphertext

$$GYHNYAKZ$$

   is written in a 26-letter alphabet ($A = 0, \cdots, Z = 25$). By applying the deciphering function

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 15 & 3 \\ 19 & 24 \end{pmatrix}$$

   to pairs of letters, determine the *first four* letters of plaintext.

2. (a) List the prime divisors of $n = 12740$.

   (b) Determine the number of integers $1 \leq k \leq 12740$ such that $gcd(k, 12740) = 1$.

   (c) Decipher the encrypted message

$$JAM$$

   which was sent through an RSA system whose deciphering key is $(n = 1147, d = 7)$; the correspondence between short message units over the 26-letter alphabet $A = 0, \ldots Z = 25$ and integers is illustrated by $YES \longrightarrow 24 \times 26^2 + 4 \times 26 + 18 = 16346$. (Hint: $920^7 = 352$ mod 1147)

3. (a) Find the inverse of the matrix

$$A = \begin{pmatrix} 3 & 2 & 6 \\ 4 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix}.$$

(b) A brewery produces Brown Ale, Dark Brown Ale and Porter. The following table summarizes the amount of malt, hops and yeast used to produce one keg of beer together with the total amount of these resources available per day.

Resource	Brown Ale	Dark Brown Ale	Porter	Daily available
Yeast	3 kg	2 kg	6 kg	250 kg
Hops	4 kg	3 kg	4 kg	220 kg
Malt	4 kg	2 kg	3 kg	170 kg

i. Let x, y, z be the number of kegs of Ale, Brown Ale and Porter produced daily. Write down a system of three linear equations which hold precisely when all three resources are fully used.

ii. Find the values of x,y,z which ensure that the daily supply of hops, malt and yeast are fully used.

(c) Find the vector $v = (x, y)$ that gets mapped to the vector $g(v) = (0, 3)$ under the linear transformation $g\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (2x + 5y, 1x + 3y)$.

4. (a) Suppose that $\binom{1}{2}$ is an eigenvector for the eigenvalue -4 of a 2×2 matrix A. What is $A\binom{-1}{-2}$?

(b) Find the characteristic polynomial, the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -1 & 4 & 0 \\ 0 & 3 & 0 \\ 8 & -12 & 7 \end{pmatrix}.$$

Hence (or otherwise) find a diagonal matrix D and a matrix E such that $E^{-1}AE = D$.

(c) Show that, if λ is an eigenvalue of an invertible matrix A, then λ^{-1} is an eigenvalue of the matrix A^{-1}.

(d) Find a matrix E such that EA is the result of replacing the first row of A by the sum of the first and third rows of A.