THE HOMOLOGY OF $SL_2(\mathbb{Z}[1/m])$ FOR SMALL m

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Abstract. We describe an algorithm for computing finitely many terms of a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ for G a finite index subgroup of $SL_2(\mathbb{Z}[1/m])$. An implementation of the algorithm is used to determine the integral homology groups $H_n(SL_2(\mathbb{Z}[1/m]),\mathbb{Z})$ for all integers $m \le 50, m \ne 30, 42$ and $n \ge 0$. For $m = 30$ or $m = 42$ the implementation is practical only for $n = 1, 2$.

1. Introduction

We describe an algorithm for computing finitely many terms of a free $\mathbb{Z}G$-resolution R_*^G of $\mathbb{Z}$ for G a finite index subgroup of $SL_2(\mathbb{Z}[1/m])$. The algorithm is valid, in principle, for any positive integer m and the modules R_n^G are finitely generated in each degree n. An implementation of the algorithm is available in the HAP [4] package for the GAP [7] computational algebra system and has been used to compute $H_n(SL_2(\mathbb{Z}[1/m]),\mathbb{Z})$ for all $m \le 50, m \ne 30, 42$ and all $n \ge 0$. When $m = 30$ or $m = 42$ the implementation is practical only for $n = 1, 2$. Table 1 summarizes these computations.

The homology of $H_n(SL_2(\mathbb{Z}[1/m]),\mathbb{Z})$ is known for all primes $m = p$ by work of Adem and Naffah [1] and these prime cases are thus omitted from Table 1. The table also omits cases where m is divisible by the square of a prime as these coincide with square free cases. It is known by work of Williams and Wisner [10] that the homology is a finite group with only 2-torsion and 3-torsion for $n > k + 1$ when $m = p_1p_2 \dots p_k$ is a product of k primes. In fact, it is easy to see from our algorithm that the homology is periodic of period 2 in degrees $> k + 1$. Each completed row of the table thus describes the integral homology of a group for all degrees n.

$m =$	$n =$ 1	2	3	4	5
6	0	$\mathbb{Z}_2 \oplus \mathbb{Z}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^2$
10	$\mathbb{Z}_3$	$\mathbb{Z}_4 \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4$
14	$\mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}_2)^3 \oplus (\mathbb{Z}_3)^2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^2$
15	$\mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_3)^2 \oplus \mathbb{Z}^2$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_3)^2$
21	$\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^2$	$(\mathbb{Z}_2)^3 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^3 \oplus \mathbb{Z}_3$
22	$\mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4$
26	$\mathbb{Z}_3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus (Z_3)^2 \oplus \mathbb{Z}^4$	$(\mathbb{Z}_2)^4 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_3)^2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^2$
30	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}$	−−	−−	−−
33	$\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^3 \oplus (\mathbb{Z}_3)^2 \oplus \mathbb{Z}^2$	$(\mathbb{Z}_2)^3$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^3 \oplus (\mathbb{Z}_3)^2$
34	$\mathbb{Z}_3$	$\mathbb{Z}_{16} \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4 \oplus \mathbb{Z}^2$	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4$
35	$\mathbb{Z}_4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_3)^2 \oplus \mathbb{Z}^6$	$(\mathbb{Z}_2)^5 \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_3)^2$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_3)^2$
38	$\mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^2 \oplus \mathbb{Z}^4$	$(\mathbb{Z}_2)^5 \oplus (\mathbb{Z}_3)^2$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^2$
39	$\mathbb{Z}_4$	$\mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}^6$	$(\mathbb{Z}_2)^5 \oplus (\mathbb{Z}_4)^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}_2 \oplus (\mathbb{Z}_4)^2 \oplus \mathbb{Z}_3$
42	0	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$	−−	−−	−−
46	$\mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{11}$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4 \oplus \mathbb{Z}^2$	$(\mathbb{Z}_2)^3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus (\mathbb{Z}_3)^4$

Table 1. Homology groups $H_n(SL_2(\mathbb{Z}[1/m]),\mathbb{Z})$

The algorithm works inductively by expressing a square free integer m in the form $m = pm'$ with p a prime, and using the decomposition $SL_2(\mathbb{Z}[1/m]) \cong SL_2(\mathbb{Z}[1/m']) *_{\Gamma_0(p)} SL_2(\mathbb{Z}[1/m'])$

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as an amalgamated free product of two distinct copies of $SL_2(\mathbb{Z}[1/m'])$ in $SL_2(\mathbb{Z}[1/m])$ over the congruence subgroup $\Gamma_0(p) \subset SL_2(\mathbb{Z}[1/m])$. A special case of a homological perturbation technique given in [6] (which, in turn, is based on a result of Wall [9]) is used to construct a resolution for $SL_2(\mathbb{Z}[1/m])$ from resolutions for $SL_2(\mathbb{Z}[1/m'])$ and $\Gamma_0(p)$. Although our primary interest is for integral coefficients, our implementation of the algorithm [4] can be used for arbitrary finitely generated coefficient modules.

The paper is structured as follows. In Section 2 we explain how to compute a resolution for a group acting on a tree from resolutions for the vertex and edge stabilizers. Since any amalgamated sum acts on an associated tree, this provides the inductive step of our algorithm. In Section 3 we give details of the initial step of the algorithm which requires a resolution for $SL_2(\mathbb{Z})$; in order to obtain as small a resolution as possible we use the action of $SL_2(\mathbb{Z})$ on the *cubic tree*: the infinite tree with every vertex of degree 3. In Section 4 we give details of how to implement the inductive step for $SL_2(\mathbb{Z}[1/m])$. In Section 5 we illustrate our implementation of the algorithm.

2. Resolutions for groups acting on trees

Let G be a discrete group acting on a contractible CW-space X in such a way that the action permutes cells. Each cell e in X has stabilizer group $G_e = \{g \in G : g.e = e\}$ whose elements need not stabilize e point-wise. Let $[e]$ denote the equivalence class of cells in the orbit of e under the action of G, and let $Orb(n)$ denote the set of equivalence classes of n-dimensional cells. The cellular chain complex $C_*(X)$ of X is an exact sequence of $\mathbb{Z}G$-modules with $H_0(C_*(X)) = \mathbb{Z}$ and with

$$C_n(X) = \bigoplus_{[e]\in Orb(n)} \mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathbb{Z}^e$$

where each $\mathbb{Z}^e$ is a copy of the integers endowed with an "orientation" action of G_e.

Suppose that for each class $[e]$ we are given a free $\mathbb{Z}G_e$-resolution $R_*^{G_e}$ of $\mathbb{Z}$. Proposition 4 in [6] gives a construction that inputs the non-free $\mathbb{Z}G$-resolution $C_*(X)$ together with the free $\mathbb{Z}G_e$-resolutions $R_*^{G_e}$, and outputs a free $\mathbb{Z}G$-resolution R_*^G of $\mathbb{Z}$. In subsequent sections we shall need this construction only for 1-dimensional spaces X. So from this point on we assume that X is a tree and recall the construction in this particularly easy case. The boundary homomorphism $C_1(X) \to C_0(X)$ induces a $\mathbb{Z}G$-equivariant homomorphism of $\mathbb{Z}G$-chain complexes

$$\begin{array}{ccc}
 & \downarrow \partial_{n+1} & & \downarrow \partial_{n+1} \\
\bigoplus_{[e]\in Orb(1)} (R_n^{G_e} \otimes_{\mathbb{Z}G_e} \mathbb{Z}^e) \otimes_{\mathbb{Z}G_e} \mathbb{Z}G & \xrightarrow{\delta_n} & \bigoplus_{[v]\in Orb(0)} R_n^{G_v} \otimes_{\mathbb{Z}G_e} \mathbb{Z}G \\
\downarrow \partial_n & & \downarrow \partial_n \\
\bigoplus_{[e]\in Orb(1)} (R_{n-1}^{G_e} \otimes_{\mathbb{Z}G_e} \mathbb{Z}^e) \otimes_{\mathbb{Z}G_e} \mathbb{Z}G & \xrightarrow{\delta_{n-1}} & \bigoplus_{[v]\in Orb(0)} R_{n-1}^{G_v} \otimes_{\mathbb{Z}G_e} \mathbb{Z}G \\
\downarrow \partial_{n-1} & & \downarrow \partial_{n-1}
\end{array}$$

which we view as a bicomplex. The required free $\mathbb{Z}G$-resolution R_*^G is the total complex of this bicomplex. That is $R_n^G = A_{1,n-1} \oplus A_{0,n}$ where

$$A_{1,n-1} = \bigoplus_{[e]\in Orb(1)} (R_{n-1}^{G_e} \otimes_{\mathbb{Z}G_e} \mathbb{Z}^e) \ \otimes_{\mathbb{G}_e} \mathbb{Z}G, \ A_{0,n} = \bigoplus_{[v]\in Orb(0)} R_n^{G_v} \otimes_{\mathbb{G}_e} \mathbb{Z}G.$$

The boundary homomorphism is

$$d_n \colon A_{1,n-1} \oplus A_{0,n} \to A_{1,n-2} \oplus A_{0,n-1}, \ x \oplus y \mapsto \partial_{n-1}(x) \oplus (-1)^n \delta_{n-1}(x) + \partial_n(y).$$

Recall that a *contracting homotopy* on the $\mathbb{Z}G$-resolution R_*^G is a family of $\mathbb{Z}$-linear homomorphisms $h_n \colon R_n^G \to R_{n+1}^G$ satisfying

$$h_{n-1}\partial_n + \partial_n h_{n-1}$$

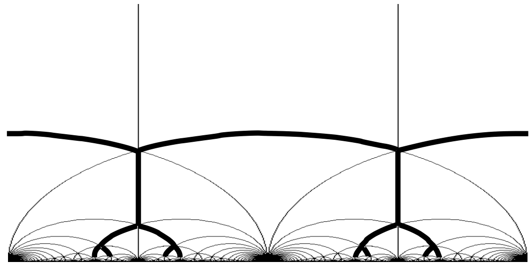

Figure 1. Portion of the cubic tree indicated in bold

for all $n \geq 0$ with $h_{-1} = 0, \partial_0 = 0$. Such a contracting homotopy can be constructed from a contracting homotopy $h'\colon C_0(X) \to C_1(X)$ and contracting homotopies $h''_n\colon R_n^{G_e} \to R_{n+1}^{G_e}$ on the $\mathbb{Z}G_e$-resolutions using the formula

$$h_n(x \oplus y) = h''_{n-1}(x) \oplus \left\{(-1)^n \partial_n \delta_n h''_{n-1}(x) + h''_n(y)\right\} \ ,$$
$$h_0(x) = h'(x) \oplus \{h''_0(x) - h''_0 \delta_0 h'(x)\} \ .$$

2.1. **A special case.** Suppose that there were just one orbit of edges represented by edge e, and that there were two orbits of vertices represented by v, v'. Suppose further that $G_v \cong G_{v'}$ and $G_e \subseteq G_v$. Then there is an equivalriant isomorphism $R_*^{G_v} \cong R_*^{G_{v'}}$. Furthermore, we can set $R_*^{G_e} = R_*^{G_v}$ and simply restrict the action to G_e. If the resolution $R_*^{G_v}$ happened to be periodic of period 2 in sufficiently high degrees (with contracting homotopy chosen to be periodic too) then the above construction yields a free ZG-resolution R_*^G for $G = G_v *_{G_e} G_{v'}$ which is periodic of period 2 in sufficiently high degrees. This observation is the basis for asserting that each complete row in Table 1 determines the homology of a group in all degrees.

We could use the special case of the construction, together with the standard resolution $R_*^{C_m}$ for finite cyclic groups C_m, to produce a free $\mathbb{Z}G$-resolution R_*^G for $G = SL_2(\mathbb{Z}) \cong C_4 *_{C_2} C_6$ in which $R_0^G = \mathbb{Z}G \oplus \mathbb{Z}G$ and $R_n^G = \mathbb{Z}G \oplus \mathbb{Z}G \oplus \mathbb{Z}G$, $n \geq 1$. However, for computational efficiency we prefer to use a slightly smaller resolution.

3. A smaller resolution for $SL_2(\mathbb{Z})$

Consider the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that $S^4 = (ST)^6 = 1$ and that the group $SL_2(\mathbb{Z})$ is generated by S and T. We let M denote the group with presentation $\langle s, u \mid s^2 = u^3 = 1\rangle$. Following [2] we construct the cubic tree Λ by taking the left cosets of $U = \langle u \rangle$ in M as vertices, and joining cosets xU and yU by an edge if, and only if, $x^{-1}y \in UsU$. Thus the vertex U is joined to sU, usU and u^2sU. The vertices of this tree are in one-to-one correspondence with all reduced words in s, u and u^2 that, apart from the identity, end in s. We say that a sequence of vertices $x_0U, x_1U, \ldots, x_nU$ is a *rooted path* if $x_0 = 1$ and there is an edge between x_iU and $x_{i+1}U$ for $0 \leq i \leq n-1$. A rooted path is described by a reduced word in s, u and u^2 ending in s.

The group M can be realized as the modular group of transformations $z \mapsto (mz+n)/(pz+q)$, m, n, p, q integers with $mq - np = 1$, of the upper half complex plane $\mathbb{H} = \{x + \mathbf{i}y \in \mathbb{C} : y > 0\}$ which is generated by the two transformations $a(z) = -1/z$ and $u(z) = z+1$. The cubic tree can thus be embedded in $\mathbb{H}$. Figure 1 illustrates this embedding by showing a portion of the cubic tree in bold against the tessellation of $\mathbb{H}$ by triangular fundamental domains for the modular group. The surjection $SL_2(\mathbb{Z}) \to M, S \mapsto s, T \mapsto s^{-1}u$ and left multiplication in M yield an action of $SL_2(\mathbb{Z})$ on the cubic tree Λ. Under this action there is one orbit of vertices and one orbit of edges.

The stabilizer group of any vertex is conjugate to $C_6 = \langle ST \rangle$. The stabilizer group of any edge is conjugate to $C_4 = \langle S \rangle$. The cellular chain complex $C_*(\Lambda)$ thus has the form

$$C_*(\Lambda): \quad \mathbb{Z}[SL_2(\mathbb{Z})] \otimes_{\mathbb{Z}C_4} \mathbb{Z}^e \stackrel{\delta}{\longrightarrow} \mathbb{Z}[SL_2(\mathbb{Z})] \otimes_{\mathbb{Z}C_6} \mathbb{Z}$$

where C_6 acts trivially on $\mathbb{Z}$ and where $\mathbb{Z}^e$ denotes the integers with non-trivial action of C_4. The boundary homomorphism δ is induced by the equivariant homomorphism of free modules

$$\mathbb{Z}[SL_2(\mathbb{Z})] \longrightarrow \mathbb{Z}[SL_2(\mathbb{Z})], 1 \mapsto T - 1 \; .$$

The construction of Section 2 can be applied to the chain complex $C_*(\Lambda)$ and standard minimal resolutions $R_*^{C_4}$, $R_*^{C_6}$ of the cyclic groups C_4, C_6 to obtain a free $\mathbb{Z}[SL_2(\mathbb{Z})]$-resoluton $R_*^{SL_2(\mathbb{Z})}$ of $\mathbb{Z}$ involving one free generator in degree 0, and two free generators in each degree ≥ 1.

The default algorithm in HAP [4] for producing resolutions of finite groups provides contracting homotopies on $R_*^{C_4}$, $R_*^{C_6}$. To obtain a contracting homotopy on $R_*^{SL_2(\mathbb{Z})}$ using the formulae in Section 2 we, additionally, need to provide a contracting homotopy on $C_*(\Lambda)$. This amounts to providing an algorithm for assigning to each vertex v the rooted path ending at v. We represent a vertex v by a matrix $A \in SL_2(\mathbb{Z})$ under the action of which v is the image of the root vertex. Thus a contracting homotopy on $C_*(\Lambda)$ boild down to an algorithm for expressing A as a reduced word in S and powers of ST.

Such an expression can be obtained using an algorithm, which we learned from [3], for expressing A as a word in S and T. It is an adaption of the Euclidean Algorithm and is best described by means of the following example taken from [3]. Suppose that we wish to express

$$A = \begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix}$$

in terms of S and T. Note that $17 = 2 \cdot 7 + 3$ and so

$$T^{-2}A = \begin{pmatrix} 3 & 5 \\ 7 & 12 \end{pmatrix}, \quad ST^{-2}A = \begin{pmatrix} -7 & -12 \\ 3 & 5 \end{pmatrix}.$$

Now note that $-7 = (-3)3 + 2$ and so

$$T^3ST^{-2}A = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad ST^3ST^{-2}A = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}.$$

Since $-3 = (-2)2 + 1$ we write

$$T^2ST^2ST^3ST^{-2}A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad ST^2ST^2ST^3ST^{-2}A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Thus $ST^2ST^2ST^3ST^{-2}A = -T = S^2T$ and $A = T^2ST^{-3}ST^{-2}ST^{-2}ST$. Using the idenitities $S^4 = 1$ and $T^{-1} = (ST)^5S$ we can now obtain an expression for A as a reduced word in S and powers of ST.

4. A resolution for $SL_2(\mathbb{Z}[1/m])$

Let m' be a square free integer, let p be a prime which is coprime to m' and set $m = pm'$. Let $\Gamma_0^{m'}(p) \subset SL_2(\mathbb{Z}[1/m'])$ be the subgroup defined by

$$\Gamma_0^{m'}(p) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; : \; c \equiv 0 \; mod \; p \}.$$

In addition to the natural inclusion we have the injection $\Gamma_0^{m'}(p) \mapsto SL_2(\mathbb{Z}[1/m'])$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix}.$$

Using these two embeddings one has an isomorphism [8]

$$SL_2(\mathbb{Z}[1/(m)]) \cong SL_2(\mathbb{Z}[1/m']) *_{\Gamma_0^{m'}(p)} SL_2(\mathbb{Z}[1/m']).$$

We can thus construct a free $\mathbb{Z}SL_2(\mathbb{Z}[1/m])$-resolution $R_*^{SL_2(\mathbb{Z}[1/m])}$ from a free $\mathbb{Z}SL_2(\mathbb{Z}[1/m'])$-resolution $R_*^{SL_2(\mathbb{Z}[1/m'])}$ using the special case 2.1 of the construction described in Section 2.

In principle one can apply this technique recursively to obtain a free $\mathbb{Z}SL_2(\mathbb{Z}[1/m])$-resolution for any m. For the recursion to work we need an algorithm for a contracting homotopy on the tree associated to the amalgamated sum. Such an algorithm boils down to one for expressing an arbitrary matrix $A \in SL_2(\mathbb{Z}[1/m])$ as a product of generators of each of the two copies of $SL_2(\mathbb{Z}[1/m'])$. The required algorithm is a slight variant of the Euclidean type algorithm from [3] described above.

5. An implementation

The user interface to our implementation of the above algorithm is illustrated by the following GAP session which computes $H_5(G,\mathbb{Z}) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_6$ for $G = \Gamma_0^2(3) \subset SL_2(\mathbb{Z}[1/2])$.

```
gap> R:=ResolutionSL2Z(2,6);
Resolution of length 6 in characteristic 0 for SL(2,Z[1/2]) .

gap> G:=CongruenceSubgroup(2,3);
CongruenceSubgroup of SL(2,Z[1/2]) level 3

gap> S:=ResolutionSubgroup(R,G);
Resolution of length 6 in characteristic 0 for CongruenceSubgroup
of SL(2,Z[1/2]) level 3 .

gap> C:=TensorWithIntegers(S);
Chain complex of length 6 in characteristic 0 .

gap> Homology(C,5);
[ 6, 6 ]
```

The two main bottlenecks in our algorithm seem to be: the $\mathbb{Z}G$-rank of R_n^G in higher degrees n; the length of the boundary $\partial(e^n)$ of certain generators of R_n^G in higher degrees.

To illustrate the first bottleneck, consider the group $G = SL_2(\mathbb{Z}[1/(2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13)]$. Our current implementation attempts to build a resolution R_*^G in which R_n^G has $\mathbb{Z}G$-rank 1075200 for all $n \geq 7$.

To illustrate the second bottleneck, consider the group $G = SL_2(\mathbb{Z}[1/30])$. Our implementation attempts to build a resolution R_*^G in which R_n^G has $\mathbb{Z}G$-rank 480 in degrees $n \geq 4$. However, one generator f of R_4^G has boundary $\partial_4(f)$ involving 1359551 $\mathbb{Z}$-free generators of R_3^G. The implementation only succeeds in computing the boundaries of 292 $\mathbb{Z}G$-free generators of R_4^G before running out of memory on a Linux PC with 16 GB of random access memory.

Some attempts have been made at applying higher-dimensional Tietze type reductions to R_*^G to overcome these bottlenecks. So far the attempts have had limited success.

6. Acknowledgements

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