Polytopal resolutions for finite groups

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Abstract

For a finite group $G$ acting faithfully on euclidean space we consider the convex hull of the orbit of a suitable vector. We show that the combinatorial structure of this polytope determines a polynomial growth free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. A resolution due to De Concini and Salvetti is recovered when $G$ is a finite reflection group. A resolution based on the simplex is obtained from the regular representation of a finite group.

Our aim in this paper is to explain how, for any finite group $G$, a finite calculation involving convex hulls leads to an explicit recursive description of all dimensions of a free $\mathbb{Z}G$-resolution in which the number of generators grows polynomially with dimension.

Let $\alpha: G \to GL(\mathbb{R}^n)$ be a faithful representation of a finite group $G$. Let $v \in \mathbb{R}^n$ be a point such that $\alpha(g)v \neq v$ for all $1 \neq g \in G$. Such a point is said to be in \textit{general position} and exists since $F(g) = \{ w \in \mathbb{R}^n : \alpha(g)w = w \}$ is a vector space of dimension less than $n$ for any $1 \neq g \in G$. Take $v$ to be any point contained in no subspace $F(g)$ for $1 \neq g \in G$. We define $P(G) = P(G, \alpha, v)$ to be the convex hull of the points in the orbit $v^G = \{ \alpha(g)v : g \in G \}$. The face lattice of $P(G)$ depends in general on the choice of $v$ as well as $\alpha$. This lattice is described for a range of groups in a forthcoming paper [2].

\textbf{Example 1.} There is an orthogonal action of the alternating group $A_4$ on $\mathbb{R}^4$ given by $\pi(w_1, w_2, w_3, w_4) = (w_{\pi^{-1}(1)}, w_{\pi^{-1}(2)}, w_{\pi^{-1}(3)}, w_{\pi^{-1}(4)})$ for $\pi \in A_4$. The vector $v = (1, 2, 3, 4)$ is in general position. Its orbit lies in the 3-dimensional hyperplane defined by the equation $\sum w_i = 10$, and the resulting 3-dimensional polytope is the icosahedron of Figure 1. \hfill $\square$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{icosahedron.png}
\caption{Icosahedron}
\end{figure}

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Recall that the *Cayley graph* of a group $G$ with respect to a set of generators $\bar{x}$ is the directed graph with vertex set $G$ and one edge from $g$ to $gx$ for each $g \in G, x \in \bar{x}$. If some generator is of order 2 then the Cayley graph will contain circuits of length 2. We define the *collapsed Cayley graph* to be that obtained from the Cayley graph by replacing, for each circuit of length two, the pair of edges by a single undirected edge.

For a faithful representation of a finite group $G$ the vertices of the polytope $P(G)$ are in one-one correspondence with the elements of $G$. The 1-skeleton of $P(G)$ can be viewed as the collapsed Cayley graph of $G$ with respect to a suitable generating set $\bar{x}_\alpha$. To specify these generators note that the vertex $v$ in $P(G)$ is connected by an edge to vertices $\alpha(g_1)v, \cdots, \alpha(g_t)v$ say. Then $\bar{x}_\alpha$ is any subset of $\{g_1, \cdots, g_t\}$ that is maximal with respect to the property that an element $g$ and its inverse $g^{-1}$ are both in $\bar{x}_\alpha$ only if $g = g^{-1}$. Since the 2-skeleton of $P(G)$ is simply connected one in fact obtains a presentation of $G$ on the generators $\bar{x}_\alpha$ with one relator for each generator of order two and one relator for each 2-face touching $v$. For instance, Figure 1 yields the presentation $A_4 = \langle x, y, z : y^2, y^3, x^3, yzx \rangle$.

We want to explain how the face lattice of the polytope $P = P(G, \alpha, v)$ determines a polynomial growth free $ZG$–resolution $F^G_\ast \to Z$. To this end we note that the representation $\alpha$ provides an action of $G$ on the polytope which permutes faces. The cellular chain complex of the polytope $C_\ast(P)$ is thus a complex of $ZG$–modules. (The abelian group $C_k(P)$ is free on generators $\bar{e}$ corresponding to $k$–cells $e$ in $P$. If $\alpha(g)e = f$ for some $g \in G$ then the module action is $g \cdot e = \pm f$ with sign depending on the orientation with which $\alpha(g)$ maps $e$ to $f$.) Moreover, the chain complex is a $ZG$–resolution of the integers since $P$ is contractible. The $ZG$–module $C_0(P)$ is free. However, the modules $C_i(P)$ are generally not free for $1 \leq i \leq n$ where $n$ denotes the dimension of $P$. We have $C_i(P) = 0$ for $i > n$. The module $C_n(P)$ is the free abelian group of rank one endowed with some possibly non-trivial action of $G$. We denote the module $C_n(P)$ by $Z^e$. The action of $G$ on $P$ permutes faces in each dimension. Associated to each $i$–dimensional face $e$ in $P$ is the stabilizer subgroup $G_e = \{ g \in G : \alpha(g)e = e \}$. We wish to give a recursive description of the resolution $F^G_\ast$ in terms of resolutions $F^G_\ast e$ for the stabilizer subgroups. Clearly $G_e$ is a proper subgroup of $G$ when $e$ has dimension $< n$. However, a minor difficulty for the recursion is that the unique $n$–dimensional cell $e$ has stabilizer group $G_e = G$. To overcome this problem we construct an infinite dimensional resolution $\hat{C}_\ast \to Z$ by splicing together countably many copies of $C_\ast(P)$. More precisely, set $C_\ast = C_\ast(P)$ and $C^e_\ast = C_\ast(P) \otimes Z^e$. Then $\hat{C}_\ast$ is periodic of the form

$$\cdots \to C_1 \to C_0 \to C^{-e}_{n-1} \to \cdots \to C^e_1 \to C^e_0 \to C^{-e}_{n-1} \to \cdots \to C_1 \to C_0.$$ 

Explicitly,

$$\hat{C}_i = \begin{cases} 
0 & i < 0, \\
C_k(P) & i = k + mn \geq 0 \text{ with } 0 \leq k < n, \text{ even } m, \\
C_k(P) \otimes Z^e & i = k + mn \geq 0 \text{ with } 0 \leq k < n, \text{ odd } m.
\end{cases}$$

The boundary map $d_i : \hat{C}_i \to \hat{C}_{i-1}$ is the map induced from $C_\ast(P)$ for all $i$ not a multiple of $n$. If $i$ is an odd multiple of $n$ then the boundary map is the composite

$$\hat{C}_i = C_0 \otimes Z^e \to Z \otimes Z^e \cong C_n(P) \to C_{n-1}(P) = \hat{C}_{i-1}.$$ 

If $i$ is an even multiple of $n$ then the boundary map is the composite

$$\hat{C}_i = C_0 \to Z \cong C_n(P) \otimes Z^e \to C_{n-1}(P) \otimes Z^e = \hat{C}_{i-1}.$$ 

By construction $\hat{C}_\ast$ is a periodic $ZG$–resolution of $Z$. Each $\hat{C}_i$ is a free abelian group on which $G$ acts by permuting free generators up to sign.
Let $[e]$ denote the orbit of a face $e$ in $P$ under the action of $G$. Let $\text{Orb}(k)$ be the set of orbits of the $k$–dimensional faces. The module $C_k(P)$ is a direct sum of $\mathbb{Z}G$–modules

$$C_k(P) = \bigoplus_{[e] \in \text{Orb}(k)} \mathbb{Z}G \otimes_{\mathbb{Z}G_e} \mathbb{Z}^e$$

(where each $\mathbb{Z}^e$ is a copy of the integers endowed with the appropriate action of $G_e$). A recursive description of the chain complex $F^G_*$ can be given by assuming that we have already constructed a free $\mathbb{Z}G_e$–resolution $F^G_e$ of $\mathbb{Z}$ for all proper subgroups $G_e$ of $G$. (In the proof of Theorem 2 we shall assume that $F^G_{2*}$ has been constructed using the action of $G_e$ on the convex hull of a subset of the vertices of $e$.) We set

$$F^G_* = F^G_{*e} \otimes_{\mathbb{Z}} \mathbb{Z}^e$$

and for $0 \leq k < n$ we set

$$D_{kj} = \bigoplus_{[e] \in \text{Orb}(k)} (\mathbb{Z}G \otimes_{\mathbb{Z}G_e} F^G_{J*})$$

to obtain a free $\mathbb{Z}G$–resolution

$$D_{k*}: \cdots \rightarrow D_{k2} \rightarrow D_{k1} \rightarrow D_{k0}$$

of the module $C_k(P)$. Setting

$$\hat{D}_{i*} = \begin{cases} 
D_{k*} & \text{if } i = k + mn, 0 \leq k < n, n \text{ even}, \\
D_{k*} \otimes_{\mathbb{Z}} \mathbb{Z}^e & \text{if } i = k + mn, 0 \leq k < n, n \text{ odd}
\end{cases}$$

we obtain a free $\mathbb{Z}G$–resolution $\hat{D}_{i*} \rightarrow \hat{C}_i$ for all $i$.

The boundary maps in $\hat{C}_*$ induce horizontal chain maps to yield a diagram $\hat{D}_{**}$ of free $\mathbb{Z}G$–modules.

\[ \cdots \rightarrow D_{n2} \rightarrow \cdots \rightarrow D_{12} \rightarrow D_{02} \]

\[ \cdots \rightarrow D_{n1} \rightarrow \cdots \rightarrow D_{11} \rightarrow D_{01} \]

\[ \cdots \rightarrow D_{n0} \rightarrow \cdots \rightarrow D_{10} \rightarrow D_{00} \]

The vertical maps $\partial^v: D_{pq} \rightarrow D_{pq-1}$ in $\hat{D}_{**}$ clearly satisfy $\partial^v \partial^v = 0$. However, $\hat{D}_{**}$ is not in general a bicomplex because the horizontal maps $\partial^h: \hat{D}_{pq} \rightarrow \hat{D}_{p,q-1}$ do not necessarily square to zero. Nevertheless, we can construct a free $\mathbb{Z}G$–chain complex $F^G_{\text{com}}$ with $F^G_{\text{com}} = \bigoplus_{p+q = n} \hat{D}_{pq}$ and differential $\partial$ given by $\partial|_{\hat{D}_{pq}} = \partial^h + (-1)^p \partial^v + \epsilon$ where $\epsilon: F^G_{\text{com}} \rightarrow F^G_{\text{com} - 1}$ is a ‘perturbation’. The perturbation is an infinite sum of module homomorphisms $\epsilon = d_2 + d_3 + \cdots$ where $d_k: \hat{D}_{**} \rightarrow \hat{D}_{*-k, ** k-1}$. On any given summand $\hat{D}_{pq}$ only finitely many terms $d_k$ are non-zero. The existence of a suitable perturbation follows from the following generalisation of a proposition of C.T.C. Wall [7], the proof of which is left to the reader.

**Proposition 1** [7] Let $A_{pq}$ ($p, q \geq 0$) be a bigraded family of free $R$–modules for some ring $R$. Suppose that there are $R$–homomorphisms $d_0: A_{pq} \rightarrow A_{p,q-1}$ such that $(A_{**, d_0})$ is an
acyclic chain complex for each $p$. Set $C_p = H_0(A_{p,*},d_0)$ and suppose further that there are $R$-homomorphisms $\delta:C_p \to C_{p-1}$ for which $(C_*,\delta)$ is an acyclic chain complex. Then there exist $R$-homomorphisms $d_k:A_{p,q} \to A_{p-k,q+k-1}$ $(k \geq 1, p > k)$ such that

(i) $d_1$ induces $\delta$;

(ii) $\sum_{i=0}^{k} d_i d_{k-i} = 0$ for each $k$ (where $d_k$ is interpreted as zero if $q = k = 0$ or if $p < k$).

Hence $d = d_0 + d_1 + \cdots: \oplus_{p+q=n} A_{pq} \to \oplus_{p+q=n-1} A_{pq}$ is a differential.

It is useful to quantify the depth of recursion in this definition of $F_*^G$. To this end define the $P$–depth $d_P(G)$ of the group $G$ as follows. If the stabilizer subgroup $G_e$ is the trivial subgroup of $G$ for all faces $e$ in $P$ of dimension less than $n$, then $d_P(G) = 0$. Otherwise

$$d_P(G) = 1 + \max_{G_e \subset G} d_P(G_e)$$

is defined in terms of the proper stabilizer subgroups $G_e$ of $G$. The value of $d_P(G)$ clearly depends on $\alpha$ and $v$. (In the recursion we can use the restriction of $\alpha$ to $G_e$ and the same vector $v$ to define the depth $d_P(G_e)$ of the stabilizer subgroups.)

**Theorem 2** (i) The chain complex $F_*^G$ is a free $ZG$–resolution of $Z$. The module $F_*^G$ is freely generated by at most $O(k^d)$ generators where $d$ is the $P$–depth of $G$.

(ii) There is a spectral sequence

$$E^1_{pq} = \bigoplus_{[e] \in Orb(p \bmod n)} H_q(G_e,Z^{e,p}) \Rightarrow H_{p+q}(G,Z)$$

where $n$ is the dimension of the polytope $P(G)$ and $Z^{e,p}$ denotes the integers endowed with an appropriate action of $G$.

**Proof.** Since each $\tilde{D}_{ij}$ is a free $ZG$–module it follows that each $F_*^G$ is also $ZG$–free. The filtration on $F_*^G$ arising from the filtration by columns of $\tilde{D}_{**}$ yields a spectral sequence with $E^1_{pq} = H_q(\tilde{D}_{p,*})$. This spectral sequence together with the exactness of each column in $\tilde{D}_{**}$ implies, as in [7], that the complex $F_*^G$ is a $ZG$–resolution of $Z$. The estimate on the number of generators is proved by induction on $d$. The spectral sequence in the theorem arises from the induced filtration on $F_*^G \otimes ZG$. We set $Z^{e,p} = Z^e$ if $p = k + mn, 0 \leq k < n$ with $m$ even. We set $Z^{e,p} = Z^e \otimes \mathbb{Z}$ if $p = k + mn, 0 \leq k < n$ with $m$ odd. □

**Example 2.** The alternating group $G = A_4$ has $P$–depth $d_P(A_4) = 1$ for the representation $\alpha$ and vector $v$ of Example 1 since, from Figure 1, we see that all stabilizer groups are either trivial or cyclic. The resolution $F_*^{A_4}$ is constructed from a diagram $\{\tilde{D}_{pq}\}_{p,q \geq 0}$ of the following form.
Each row in $\bar{D}_*$ is periodic of period 4 (since $A_4$ acts trivially on $C_3(P(A_4))$). The horizontal maps are all zero in the second and higher rows. The maps in the bottom row are induced by the boundary in $C_3(P(A_4))$. The columns are equal to either $ZG$, or a direct sum of $ZG$ with a periodic resolution of period 2, or a direct sum of $(ZG)^2$ with a periodic resolution of period 2. We have $F_n^{A_4} = \oplus_{p+q=n} D_{pq}$ and the boundary map in the resolution is determined by the vertical maps $d_0$, the horizontal maps $d_1$ and the perturbations $d_k$ ($k \geq 2$).

**Example 3.** Let $W$ be a finite subgroup of $O(\mathbb{R}^n)$ generated by reflections, and such that $W$ fixes no non-zero vector in $\mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is in general position for the inclusion $\alpha: W \rightarrow O(\mathbb{R}^n)$ if and only if it does not lie in a hyperplane corresponding to a reflection in $W$. It is readily seen that the polytope $P(W) = P(W, \alpha, v)$ has the same combinatorial structure for any general position vector $[2]$. Moreover, the generating set $\varphi_n$ defined above is a set of $n$ simple reflections for $W$. The $P$-depth of $W$ is equal to $n - 1$, and the orbits $[\alpha]$ of the $k$-dimensional faces of $P(W)$ correspond to the subsets $\varphi_n$ of $\varphi_n$ of size $k$. The vertices of a $k$-dimensional facet $\alpha$ are the points $g \cdot v$ where $g$ ranges over the finite reflection group generated by $\varphi_n$. The stabilizer $G_{\alpha}$ is the subgroup generated by $\varphi_n$. A straightforward induction shows that the free generators of the $ZW$-module $F^W_i$ are in one-one correspondence with the multisets of size $i$ over $\varphi_n$. An explicit description of the boundary map is given in [4] where it is used to derive a simple formula for the integral homology of $W$ in dimensions $\leq 3$.

This resolution for finite reflection groups $W$ is originally due to C. De Concini and M. Salvetti [1] who obtained it using different methods.

**Example 4.** The regular representation of a finite group $G$ is an action on $\mathbb{R}^n$ with $n = |G|$. The elements $g \in G$ are associated to distinct standard basis vectors $v_g$, and the action is given by $g'v_g = v_{g'g}$. Any standard basis vector $v$ is in general position, and the resulting polytope $P(G)$ is an $(n-1)$-dimensional simplex. In this case the spectral sequence of Theorem 2 has the same abelian groups in the $E^1$ page as that of Proposition 5.2 in [5].

The following shows that the resolution $F_*^G$ could be easily implemented on a computer once the combinatorial structure of $P(G, \alpha, v)$ is known.

**Theorem 3** The generators and boundary map of the resolution $F_*^G$ are described by explicit recursive formulae based on a finite table of data derived from the combinatorial structure of the polytope $P(G, \alpha, v)$.

**Proof.** The theorem follows from induction on the $P$-depth of $G$. If the $P$-depth is zero then we are done. Otherwise, for each proper stabilizer subgroup $G_{\alpha}$ in $G$ we suppose that the resolution $F_*^{G_{\alpha}}$ is determined by a finite table of data, and that there are suitable ‘contracting homomorphisms’ $h$ for $F_*^{G_{\alpha}}$. The following proposition completes the induction; one could use [6, Lemma 2] for its last assertion.

**Proposition 4** Let the family of modules $A_*$ and module homomorphisms $d_0: A_{p, q} \rightarrow A_{p, q-1}$ and $d: C_p \rightarrow C_{p-1}$ be as in Proposition 1. Suppose that there exist abelian group homomorphisms $h_0: A_{p, q} \rightarrow A_{p, q+1}$ such that $d_0 h_0 d_0(x) = d_0(x)$ for all $x \in A_{p, q+1}$. Then we can construct the module homomorphisms $d_k: A_{p, q} \rightarrow A_{p-k, q+k-1}$ and differential $d$ of Proposition 1 by first lifting $d$ to $d_1: A_{p, 0} \rightarrow A_{p-1, 0}$ and then recursively defining $d_k = -h(\sum_{i=1}^k d_i d_{k-i})$ on free generators of the module $A_{p, q}$. Furthermore, if $H_0(C_*) \cong \mathbb{Z}$ and each $C_p$ is free abelian, we can construct abelian group homomorphisms $h_*: \oplus_{p+q=n} A_{pq} \rightarrow \oplus_{p+q=n+1} A_{pq}$ satisfying $dh_*(x) = d(x)$ by setting $h_* (a_{pq}) = h_0(a_{pq}) - h_0 h_0(a_{pq}) + \varepsilon(a_{pq})$ for free generators $a_{pq}$ of the abelian group $A_{pq}$. Here $d^* = \sum_{i=1}^q d_i$ and, for $q \geq 1$, $\varepsilon = 0$. For $q = 0$ we define $\varepsilon = h_1 - h_0 h_0 h_1 + h^* h_0 h^* h_1$ where $h_1: A_{p, 0} \rightarrow A_{p+1, 0}$ is an abelian group homomorphism induced by a contracting homotopy on $C_*$. 


The technique underlying Theorem 2 can be applied to certain infinite groups of isometries of euclidean or hyperbolic space. Resolutions for infinite generalized triangle groups can, for instance, be obtained in this way [3].

References


