Derived functors and the homology of n-types

J.M. Casas *

Dpto. Matémática Aplicada, Univ. de Vigo, E.U.I.T. Forestal, 36005, Pontevedra, Spain

G. Ellis

Mathematics Department, National University of Ireland Galway, Galway, Ireland

M. Ladra *

Dpto. de Alxebra, Universidad de Santiago, Santiago de Compostela, Spain

T. Pirashvili

A. Razmadze Mathematical Intitute, Aleksidze str. 1, Tbilisi, Republic of Georgia

1 Introduction

Let X be a connected CW-space whose homotopy groups $\pi_i X$ are trivial in dimensions $i > n+1$. Such a space is termed a *homotopy* $(n+1)$*-type*. In the case $n = 0$, classical homological algebra provides a purely algebraic description of the integral homology $H_*(X)$ in terms of derived functors. For $n = 1$ it has recently been shown [8] (*cf.* [2]) that the homology can be realised as the nonabelian left derived functors of a certain abelianisation functor $\mathcal{A}$: (crossed modules) $\to$ (abelian groups). Crossed modules are convenient algebraic models of homotopy 2-types. More generally, homotopy $(n+1)$-types are modelled by cat^n-groups [9] or equivalently by crossed n-cubes [6]. Our aim in this paper is to explain how the methods of [8] extend to arbitrary $n \geq 0$ and lead to a natural isomorphism

$$H_{n+i+1}(X) \cong L_i^{\mathcal{A}}(G),\ i \geq 1 \tag{1}$$

where $L_i^{\mathcal{A}}(-)$ is the i-th nonabelian left derived functor of a certain abelianisation functor $\mathcal{A}$: (crossed n-cubes) $\to$ (abelian groups) and G is a suitable crossed n-cube. We also explain how the relationship between $H_{n+1}(X)$ and $L_0^{\mathcal{A}}(G)$ can be expressed as an algebraic formula for the homology of X analogous to the Hopf type formula for the higher homology of a group obtained in [1].

The paper handles only the cases $n = 1, 2$ in full detail. The routine modifications needed for $n \geq 3$ are largely left to the reader. In Section 2 we recall some terminology and results of D. Quillen [12] [14] on homology in algebraic categories. In Section 3 we derive the following

*The first and third authors were supported by Xunta de Galicia, grant PGIDT00PX37101PR and by MCYT, project BFM 2000-0523, Spain. The second and fourth authors are very grateful to the University of Santiago and the University of Vigo for hospitality.

lemma from general results on simplicial objects: if G is a projective n-fold simplicial group then $\pi_0(G)$ is a free group and $\pi_i(G) = 0$ for $i \geq 1$. Isomorphism (1) is proved in Section 4, together with various Hopf-type formulae for $H_*(X)$.

We adopt the following notation. The category of sets is denoted by $\mathcal{S}$. The category of groups is denoted by $\mathcal{G}$. The category of simplicial objects of a category $\mathcal{C}$ is denoted by $\mathcal{SC}$. Accordingly, $\mathcal{SS}$ denotes the category of simplicial sets, $\mathcal{SG}$ the category of simplicial groups. The category of n-fold simplicial groups is denoted by $\mathcal{S}^n\mathcal{G}$. We always identify an object X of $\mathcal{C}$ with the constant simplicial object of $\mathcal{SC}$ whose simplicial operators are all equal to the identity morphism of X. The free group on a set X is denoted by $< X >_{gr}$. The standard n-simplex is denoted by $\mathbf{\Delta}^n$ and characterised by the property $\mathrm{Hom}_{\mathcal{SS}}(\mathbf{\Delta}^n, X) \cong X_n$ for all simplicial sets X. The integral homology of a simplicial set or CW-space X is denoted by $H_*(X)$. For a simplicial group G, we let $M_*(G)$ be the Moore complex of G, which is the nonabelian chain complex defined by

$$M_n(G) = \bigcap_{0 \leq i < n} \mathsf{Ker}\ \partial_i^n,$$

with $d : M_n(G) \to M_{n-1}(G)$ induced by ∂_0^n.

2 Quillen homology

In this section we recall some ideas and results of Quillen (see [12] and [14]).

Let $\mathcal{C}$ be a category with finite limits. We let $X \times_Y X$ denote the pull-back

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

of a morphism $f\colon X \to Y$ in $\mathcal{C}$. The morphism $f\colon X \to Y$ is said to be an *effective epimorphism* if, for any object T, the diagram of sets

$$\mathrm{Hom}_{\mathcal{C}}(Y,T) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X,T) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{C}}(X \times_Y X, T)$$

is exact. This means that f^* is an injective map and, if $g\colon X \to T$ is a morphism such that $gp_1 = gp_2$, then there exists a (necessarily unique) morphism $h\colon Y \to T$ such that $g = hf$.

An object P of $\mathcal{C}$ is *projective* if for any diagram

$$\begin{array}{ccc} & & P \\ & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with f an effective epimorphism there exists a morphism $h\colon P \to X$ such that $g = fh$. We say that $\mathcal{C}$ has sufficiently many projective objects if for any object X there is a projective object P and an effective epimorphism $P \to X$.

Let us assume additionally that $\mathcal{C}$ possesses colimits. An object X is called *small* if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ commutes with filtered colimits.

A class $\mathcal{U}$ of objects of $\mathcal{C}$ is said to *generate* $\mathcal{C}$ if for every object X there is an effective epimorphism $Q \to X$ where Q is a coproduct of copies of members of $\mathcal{U}$.

The category $\mathcal{C}$ is said to be an *algebraic category* if it possesses finite limits and arbitrary colimits and has a set of small projective generators. We leave the proof of the following easy fact to the reader.

Lemma 1 *Let $\mathcal{C}$ be an algebraic category and $\mathcal{B} \subset \mathcal{C}$ be a full subcategory. Suppose that $\mathcal{B} \subset \mathcal{C}$ has a left adjoint $\mathcal{L}: \mathcal{C} \to \mathcal{B}$ and that the following condition holds: a morphism $f: X \to Y$ in $\mathcal{B}$ is an effective epimorphism in $\mathcal{C}$ if and only if it is an effective epimorphism in $\mathcal{B}$. Then $\mathcal{L}P$ is a projective object in $\mathcal{B}$ for each projective object P in $\mathcal{C}$. Thus $\mathcal{B}$ is an algebraic category.*

The following fundamental result is due to Quillen (see Theorem 4 in [12], Chapter II, page 4.).

Theorem 2 *Let $\mathcal{C}$ be an algebraic category. Then there exists a unique closed simplicial model category structure on the category $\mathcal{SC}$ of simplicial objects over $\mathcal{C}$ such that a morphism f in $\mathcal{SC}$ is a fibration (resp. weak equivalence) if and only if $\mathrm{Hom}_{\mathcal{SC}}(P, f)$ is a fibration (resp. weak equivalence) of simplicial sets for each projective object P of $\mathcal{C}$. Moreover, if X is a cofibrant object in $\mathcal{SC}$ then X_n is a projective object in $\mathcal{C}$ for all $n \geq 0$.*

A *simplicial resolution* of an object X of $\mathcal{C}$ can be defined as a fibration $Q \to X$ in $\mathcal{SC}$ which is also a weak equivalence. If additionally Q is a cofibrant object in $\mathcal{SC}$ then $Q \to X$ is called a simplicial *cofibrant* resolution. It is a formal consequence of Theorem 1 that simplicial cofibrant resolutions exist and are unique up to homotopy equivalence, and up to homotopy depend functorially on X.

Let $\mathcal{C}_{ab}$ denote the category of abelian group objects in the algebraic category $\mathcal{C}$. It can be shown that $\mathcal{C}_{ab}$ is an abelian category and that moreover the abelianisation functor

$$(-)_{ab}: \mathcal{C} \to \mathcal{C}_{ab}, X \mapsto X_{ab}$$

left adjoint to the forgetful functor $\mathcal{C}_{ab} \subset \mathcal{C}$, exists. Following [14] one defines the Quillen homology of an object X in $\mathcal{C}$ as the homology of the chain complex associated to the simplicial object Q_{ab} obtained by applying $(-)_{ab}$ dimensionwise to a simplicial cofibrant resolution Q of X. We let $D_*(X)$ denote the Quillen homology of X.

3 Projective objects in n-fold simplicial groups

The material in this section is well-known. Our goal is the following result: a projective object in $\mathcal{S}^n\mathcal{G}$ has no homotopy in dimensions ≥ 1.

Let $\mathcal{I}$ be a small category and let $\mathcal{G}^{\mathcal{I}}$ denote the category of functors $\mathcal{I} \to \mathcal{G}$ from $\mathcal{I}$ to the category of groups.

Lemma 3 *A morphism $f: X \to Y$ in $\mathcal{G}^{\mathcal{I}}$ is an effective epimorphism if and only if $f(i)$ is surjective for all objects $i \in \mathcal{I}$.*

Proof. Assume $f(i)$ is surjective for all objects $i \in \mathcal{I}$. Then $\mathrm{Hom}(Y, T) \to \mathrm{Hom}(X, T)$ is injective and, for any functor $T: \mathcal{I} \to \mathcal{G}$ and any natural transformation

$$g: X \to T$$

such that the diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{g} & T \end{array}$$

commutes, there exists a unique transformation

$$h: Y \to T$$

such that $g = hf$. The transformation h is given by

$$h(i)(y) = g(i)(f(i)^{-1}(y)),\ y \in Y(i)\,.$$

The condition on g implies that h is well-defined. Conversely, assume f is an effective epimorphism. Set $Y'(i) = \mathrm{Im}(f)(i) \subset Y(i)$. Then each $X(i) \to Y'(i)$ is surjective and hence $X \to Y'$ is an effective epimorphism. Therefore $\mathrm{Hom}(Y, Z)$ and $\mathrm{Hom}(Y', Z)$ are both equalisers of the same diagram and hence coincide. It follows that $Y' = Y$. □

Lemma 4 *For each object $i \in \mathcal{I}$ let*

$$h_i: \mathcal{I} \to \mathcal{G}$$

be the functor given by

$$j \mapsto < \mathrm{Hom}_{\mathcal{I}}(i, j) >_{gr}\ .$$

Then the collection $(h_i)_{i \in \mathcal{I}}$ is a set of small projective generators in the category $\mathcal{G}^{\mathcal{I}}$.

Proof. The Yoneda lemma implies that

$$\mathrm{Hom}_{\mathcal{G}^{\mathcal{I}}}(h_i, T) \cong T(i)$$

for $i \in \mathcal{I}, T \in \mathcal{G}^{\mathcal{I}}$ and the result follows. □

Corollary 5 *The category $\mathcal{G}^{\mathcal{I}}$ is an algebraic category.*

In fact, one can prove that $\mathcal{C}^{\mathcal{I}}$ is an algebraic category for any algebraic category $\mathcal{C}$. As a consequence of Corollary 5 we see that the category of (n-fold) simplicial groups is algebraic. We need to identify the homotopy type of projective objects in $\mathcal{S}^n\mathcal{G}$.

Let $\mathbf{\Delta}$ be the category of finite ordinals. We will assume that the objects are the sets

$$[n] = \{0, 1, \cdots, n\},\ n \geq 0$$

and morphisms are nondecreasing maps. Then $\mathcal{G}^{\mathbf{\Delta}^{op}}$ is the category of simplicial groups. Since

$$\mathbf{\Delta}^n = \mathrm{Hom}_{\mathbf{\Delta}^{op}}([n], -)$$

is the standard n-simplex, Lemma 4 shows that any projective object in the category of simplicial groups is a retract of a simplicial group of the form

$$< \coprod_{n \geq 0} (S_n \times \mathbf{\Delta}^n) >_{gr}\,,$$

where $(S_n)_{n \geq 0}$ is a sequence of sets.

It is well-known that the standard n-simplex is simplicially contractible (see for example ρ on page 151 of [7]) and therefore the projection of $\sqcup_{n\geq 0} S_n \times \mathbf{\Delta}^n$ to the constant simplicial set $\sqcup_{n\geq 0} S_n$ is a simplicial homotopy equivalence. Since any degreewise extension of a functor $\mathcal{S} \to \mathcal{G}$ preserves the homotopy relation, we see that the simplicial group $< \sqcup_{n\geq 0} S_n \times \mathbf{\Delta}^n >_{gr}$ is homotopy equivalent to the constant simplicial group $< \sqcup_{n\geq 0} S_n >_{gr}$. As a consequence we obtain the following.

Corollary 6 *If P is a projective object in the category of simplicial groups, then P is degreewise free. Moreover $\pi_0 P$ is a free group and P is homotopy equivalent to a constant simplicial group, hence $\pi_i P = 0$ for $i \geq 1$.*

We now consider bisimplicial groups. Given a bisimplicial group G we let G_n^v and G_m^h denote the n-th vertical and m-th horizontal parts; both are simplicial groups. We let $\pi_i^v G$ and $\pi_i^h G$ denote the simplicial groups obtained by taking the i-th homotopy group of G_n^v and G_m^h. We let $\pi_i G$ denote the i-th homotopy group of the diagonal simplicial group $(G_{n,n})_{n\geq 0}$. For simplicial sets X and Y we let $X \boxtimes Y$ denote the bisimplicial set whose (m,n)-th component is $X_m \times Y_n$.

By Corollary 5 the category of bisimplicial groups is an algebraic category and any projective object is a retract of a bisimplicial group of the form

$$< \bigsqcup_{n,m} S_{n,m} \times (\mathbf{\Delta}^n \boxtimes \mathbf{\Delta}^m) >_{gr} ,$$

where $(S_{n,m})_{n,m\geq 0}$ is a family of sets. Thus each vertical or horizontal part is a projective object in the category of simplicial groups. Moreover, if P is a projective object in the category of bisimplicial groups, then

$$\pi_n^v P = 0 = \pi_n^h P, \ n \geq 1$$

and both $\pi_0^v P$ and $\pi_0^h P$ are projective objects in the category of simplicial groups. Thus

$$\pi_i^h \pi_j^v P = 0$$

as soon as $i > 0$ or $j > 0$. So the spectral sequence [13]

$$E_{pq}^2 = \pi_p^v \pi_q^h P \Rightarrow \pi_{p+q} P$$

implies that $\pi_i P = 0$ for $i \geq 1$. Since $\pi_0 P = \pi_0^v \pi_0^h P$ we see that $\pi_0 P$ is a free group.

The situation for multisimplicial groups is analogous. We leave as an exercise to the reader the modifications required to obtain the following result.

Lemma 7 *Let G be a projective object in the category $\mathcal{S}^n\mathcal{G}$ of n-fold simplicial groups. In each direction G is homotopy equivalent to a constant simplicial object in the category $\mathcal{S}^{n-1}\mathcal{G}$ of $(n-1)$-fold simplicial groups, which is also a projective in $\mathcal{S}^{n-1}\mathcal{G}$. In particular, $\pi_i G = 0$ for $i \geq 1$, and $\pi_0 P$ is a free group, where $\pi_i G$ denotes the i-th homotopy group of the diagonal of G.*

4 Homology of cat^n-groups and crossed n-cubes

Recall [9] that a *cat^n-group* consists of a group G together with $2n$ endomorphisms $s_i, t_i\colon G \to G$ satisfying

$s_i s_i = s_i, s_i t_i = t_i, t_i t_i = t_i, t_i s_i = s_i,$
$s_i t_j = t_j s_i \ (i \neq j),$
$[\mathsf{Ker}(s_i), \mathsf{Ker}(t_i)] = 0,$

for $1 \leq i \leq n$. A *morphism* of cat^n-groups $(G, s_i, t_i) \to (G', s'_i, t'_i)$ is a group homomorphism $G \to G'$ that preserves the s_i and t_i. We let $\mathcal{CG}$ denote the category of cat^1-groups, and $\mathcal{C}^n\mathcal{G}$ the category of cat^n-groups. Note that a cat^0-group is just a group.

A cat^1-group (G, s, t) is equivalent to a category object in $\mathcal{G}$. The arrows are the elements of G, the identity arrows are the elements of $N = \mathsf{Im}(s) = \mathsf{Im}(t)$, the source and target maps are s and t, and composition of arrows $g, h \in G$ is given by $g \circ h = g(sg)^{-1}h$. Thus the nerve of a category provides a functor $\mathcal{N}\colon \mathcal{CG} \to \mathcal{SG}$.

Lemma 8 i) *$\mathcal{N}$ is a full and faithful;*

ii) *$\mathcal{N}$ possesses a left adjoint $\mathcal{T}\colon \mathcal{SG} \to \mathcal{CG}$ and the functor $\mathcal{N} \circ \mathcal{T} : \mathcal{SG} \to \mathcal{SG}$ preserves the homotopy relation. Moreover for any simplicial group G one has*

$$\pi_i(\mathcal{N} \circ \mathcal{T}(G)) = \pi_i(G) \text{ if } i = 0, 1$$

and

$$\pi_i(\mathcal{N} \circ \mathcal{T}(G)) = 0 \text{ if } i > 1.$$

iii) *a morphism f in $\mathcal{CG}$ is an effective epimorphism if and only if $\mathcal{N}f$ is an effective epimorphism in $\mathcal{SG}$;*

iv) *$\mathcal{CG}$ is an algebraic category. Moreover if (G, s, t) is a projective object in the category $\mathcal{CG}$ of cat^1-groups, then $\pi_i(\mathcal{N}(G)) = 0$ for $i > 0$ and $\pi_0(\mathcal{N}(G))$ is free group.*

Proof. i) is obvious. The statement ii) is well-known and it follows for example from Proposition 3 of [11]. By *loc. cit.* the cat^1-group $\mathcal{T}G$ has underlying group $G_1/\partial_0^2(M_2(G))$; the maps s and t are induced by d_0^1 and d_1^1. One easily checks that for any simplicial group G the Moore complex of $\mathcal{N} \circ \mathcal{T}(G)$ is isomorphic to

$$\cdots \to 0 \to M_1(G)/\partial_0^2(M_2(G)) \to M_0(G)$$

and therefore $\pi_i(\mathcal{N} \circ \mathcal{T}(G)) = \pi_i(G)$ if $i = 0, 1$ and $\pi_i(\mathcal{N} \circ \mathcal{T}(G)) = 0$ if $i > 1$. In order to verify iii), note that the argument given in the proof of Lemma 3 shows that f is an effective epimorphism if and only if f is surjective (as a homomorphism of groups). But f is surjective if and only if $\mathcal{N}f$ is degreewise surjective which, by Lemma 3, is equivalent to $\mathcal{N}f$ being an effective epimorphism. By iii) the assumptions of lemma 1 hold and therefore $\mathcal{CG}$ is an algebraic category. The last statement of iv) follows easily from Lemma 6 and from ii). □

A cat^2-group G is equivalent to a category object in $\mathcal{CG}$. It is thus equivalent to a group endowed with two compatible category structures, a horizontal one and a vertical one. The nerve $\mathcal{N}^v G$ of the vertical category structure is a category object in $\mathcal{SG}$. By then taking the nerve of the horizontal category structure, we obtain a full and faithful functor

$$\mathcal{N}^2 = \mathcal{N}^h \mathcal{N}^v\colon \mathcal{C}^2\mathcal{G} \to \mathcal{S}^2\mathcal{G}$$

into bisimplicial groups. Moreover a morphism f is an effective epimorphism (*i.e.* surjective as a group homomorphism) in $\mathcal{C}^2\mathcal{G}$ if and only if $\mathcal{N}^2 f$ is an effective epimorphism (*i.e.* dimensionwise surjective) in $\mathcal{S}^2\mathcal{G}$. The functor $\mathcal{N}^2$ admits a left adjoint

$$\mathcal{T}^2 \colon \mathcal{S}^2\mathcal{G} \to \mathcal{C}^2\mathcal{G}$$

which is defined by first applying $\mathcal{T}$ dimensionwise to a bisimplicial group G to obtain a simplicial cat^1-group $\mathcal{T}G$, and then applying $\mathcal{T}$ again to obtain a cat^2-group $\mathcal{T}^2 G$.

By Corollary 5 and Lemma 1 the category $\mathcal{C}^2\mathcal{G}$ of cat^2-groups is an algebraic category. Moreover if (G, s_1, s_2, t_1, t_2) is a projective object in the category $\mathcal{C}^2\mathcal{G}$ of cat^2-groups, then the horizontal and vertical cat^1-groups are projective in the category of cat^1-groups. Moreover $\pi_i(\mathcal{N}^2(G)) = 0$ for $i > 0$ and $\pi_0(\mathcal{N}^2(G))$ is free group. These facts follows easily from Lemma 7 because $\mathcal{T}$ respects the homotopy relations.

The situation for cat^n-groups is similar. We leave as an exercise for the reader the routine modifications needed to establish the following.

Lemma 9 *The category $\mathcal{C}^n\mathcal{G}$ of cat^n-groups is an algebraic category. Moreover if (G, s_i, t_i), $i = 1, \cdots, n$ is a projective object in the category $\mathcal{C}^n\mathcal{G}$, then each 'face' of G is a projective object in the category $\mathcal{C}^{n-1}\mathcal{G}$. Furthermore $\pi_i(\mathcal{N}^n(G)) = 0$ for $i > 0$ and $\pi_0(\mathcal{N}^n(G))$ is a free group.*

An abelian group object in $\mathcal{C}^n\mathcal{G}$ is just a cat^n-group whose underlying group is abelian. The abelianisation functor

$$(-)_{ab} \colon \mathcal{C}^n\mathcal{G} \to (\mathcal{C}^n\mathcal{G})_{ab}$$

sends a cat^n-group $G = (G, s_i, t_i)$ to the cat^n-group with underlying group $G_{ab} = G/[G,G]$ and induced homomorphisms $s_i, t_i \colon G_{ab} \to G_{ab}$. The Quillen homology of a cat^n-group G is obtained from a cofibrant simplicial resolution $Q \to G$ by abelianising the simplicial cat^n-group Q dimensionwise and taking the homology of the associated chain complex or associated Moore complex:

$$D_i(G) = \pi_i(Q_{ab})\,.$$

Note that $D_i(G)$ is an abelian cat^n-group for each $i \geq 0$. Below we define the group $H_i(G)_{\text{Quillen}}$ as a subgroup of the underlying group of $D_{i-1}(G)$.

There is an alternative way to define the homology of a cat^n-group G, based on the composite functor

$$\mathcal{B} \colon \mathcal{C}^n\mathcal{G} \xrightarrow{\mathcal{N}^n} \mathcal{S}^n\mathcal{G} \xrightarrow{\mathcal{N}} \mathcal{S}^{n+1}\mathcal{S} \xrightarrow{\text{diagonal}} \mathcal{S}\mathcal{S}$$

from cat^n-groups to simplicial sets ($n \geq 0$). The functor $\mathcal{N} \colon \mathcal{S}^n\mathcal{G} \to \mathcal{S}^{n+1}\mathcal{S}$ is defined by considering groups as categories and taking the nerve degreewise. The geometric realization $|\mathcal{B}G|$ is by definition the *classifying space* of the cat^n-group G and induces an equivalence between the (suitably defined) homotopy categories of cat^n-groups and connected CW-spaces X with $\pi_i X = 0$ for $i \geq n+2$ (see [9]). The integral homology of $|\mathcal{B}G|$ is a natural homology to associate to G, and so we set

$$H_i(G)_{\text{Top}} = H_i(|\mathcal{B}G|),\ i \geq 0.$$

We refer the reader to [5] and [4] for more information on $H_i(G)_{\text{Top}}$ in the case $n = 1$. Our principal aim in this paper is a comparison of the algebraically defined homology $D_*(G)$ with the topologically defined homology $H_*(G)_{\text{Top}}$.

We remark that the classifying functor $\mathcal{B}$ behaves nicely with respect to the inclusion functor $\mathcal{C}^n\mathcal{G} \to \mathcal{SC}^n\mathcal{G}$ and also with respect to the inclusion functors $\mathrm{incl}_j\colon \mathcal{C}^n\mathcal{G} \longrightarrow \mathcal{C}^{n+1}\mathcal{G}$ $(1 \le j \le n+1)$ which insert identity morphisms s_j, t_j. By taking nerves and diagonals appropriately one obtains a functor $\mathcal{B}\colon \mathcal{SC}^n\mathcal{G} \to \mathcal{SS}$ from simplicial cat^n-groups to simplicial sets such that the triangle of functors

$$\begin{array}{ccc} \mathcal{C}^n\mathcal{G} & \xrightarrow{\text{inclusion}} & \mathcal{SC}^n\mathcal{G} \\ & \searrow^{\mathcal{B}} & \downarrow{\scriptstyle \mathcal{B}} \\ & & \mathcal{SS} \end{array}$$

commutes. The triangle of functors

$$\begin{array}{ccc} \mathcal{C}^n\mathcal{G} & \xrightarrow{\mathrm{incl}_j} & \mathcal{C}^{n+1}\mathcal{G} \\ & \searrow^{\mathcal{B}} & \downarrow{\scriptstyle \mathcal{B}} \\ & & \mathcal{SS} \end{array}$$

also commutes for each j.

To facilitate the comparison of $H_*(G)_{\mathrm{Top}}$ and $H_*(G)_{\mathrm{Quillen}}$ we recall from [6] some details on the categorical equivalence between cat^n-groups and crossed n-cubes. A *crossed n-cube* consists of a collection of groups M_α indexed by the 2^n subsets $\alpha \subset \{1, \cdots, n\}$, together with homomorphisms $\lambda_i\colon M_\alpha \to M_{\alpha\setminus\{i\}}$ for $i \in \alpha$ and commutator type functions $h\colon M_\alpha \times M_\beta \to M_{\alpha\cup\beta}$. For present purposes it is unnecessary to recall precise details of the commutator functions or the axioms satisfied by the structure. A crossed 1-cube

$$M_{\{1\}} \xrightarrow{\lambda_1} M_\emptyset$$

is just a crossed module, the action being given by $M_\emptyset \times M_{\{1\}} \to M_{\{1\}}, (x, y) \mapsto h(x, y)y$. A crossed 2-cube

$$\begin{array}{ccc} M_{\{1,2\}} & \xrightarrow{\lambda_1} & M_{\{2\}} \\ \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \lambda_2} \\ M_{\{1\}} & \xrightarrow{\lambda_1} & M_\emptyset \end{array}$$

coincides with the notion of a crossed square introduced by Loday [9]. A morphism $(M_\alpha) \to (M'_\alpha)$ of crossed n-cubes is a family of structure preserving group homomorphisms $M_\alpha \to M'_\alpha$. We let $\mathcal{XG}$ denote the category of crossed modules, and $\mathcal{X}^n\mathcal{G}$ the category of crossed n-cubes.

It has long been known that a crossed module is equivalent to a category object in $\mathcal{G}$, that is, to a cat^1-group (see [3]). Loday [9] proved that crossed squares are equivalent to cat^2-groups, and this equivalence was extended [6] to one between crossed n-cubes and cat^n-groups. The functorial equivalence

$$\mathcal{E}\colon \mathcal{C}^n\mathcal{G} \to \mathcal{X}^n\mathcal{G}$$

sends a cat^n-group $G = (G, s_i, t_i)$ to the crossed n-cube $\mathcal{E}G$ with

$$\mathcal{E}G_\alpha = \bigcap_{i\in\alpha} \mathsf{Ker}\,(s_i) \cap \bigcap_{j\in\overline{\alpha}} \mathsf{Im}\,(s_j)$$

where $\overline{\alpha}$ denotes the complement of α in $\{1, \cdots, n\}$. The morphisms $\lambda_i\colon \mathcal{E}G_\alpha \to \mathcal{E}G_{\alpha\setminus\{i\}}$ are the restriction of t_i, and the functions h are all given by commutation in the group G. It is convenient to let σG denote the group

$$\sigma G = \mathcal{E}G_{\{1,\cdots,n\}} = \cap_{1\le i\le n} \mathsf{Ker}\,(s_i).$$

The inverse equivalence $\mathcal{E}^{-1}\colon \mathcal{X}^n\mathcal{G} \to \mathcal{C}^n\mathcal{G}$ is described in [6]. For a crossed n-cube M we set $\mathcal{B}M = \mathcal{B}(\mathcal{E}^{-1}M)$.

The equivalence $\mathcal{E}\colon \mathcal{CG} \to \mathcal{XG}$ induces an equivalence $\mathcal{E}\colon \mathcal{SCG} \to \mathcal{SXG}$ such that the diagram

$$\begin{array}{ccc} \mathcal{CG} & \xrightarrow{\text{inclusion}} & \mathcal{SCG} \\ \downarrow{\scriptstyle \mathcal{E}} & & \downarrow{\scriptstyle \mathcal{E}} \\ \mathcal{XG} & \xrightarrow{\text{inclusion}} & \mathcal{SXG} \end{array}$$

commutes.

We need the following easily verified description of the crossed n-cube $\mathcal{E}(G_{ab})$ associated to the abelianisation of a cat^n-group G.

Lemma 10 *Let G be a cat^n-group with associated crossed n-cube $\mathcal{E}G = (M_\alpha)$. Then the crossed n-cube associated to G_{ab} has the form $\mathcal{E}(G_{ab}) = (\overline{M}_\alpha)$ where*

$$\overline{M}_\alpha = M_\alpha / \prod_{\beta\cup\gamma=\alpha,\beta\cap\gamma=\emptyset} [M_\beta, M_\gamma],$$

commutator subgroups being defined via commutation in the underlying group of G.

The comparison of $H_*(G)_{\rm Top}$ with $D_*(G)$ is facilitated by setting

$$H_i(G)_{\rm Quillen} = \sigma D_{i-1}(G), \quad i \ge 1$$

$$H_0(G)_{\rm Quillen} = \mathbf{Z}.$$

We also denote by $\bar{H}_i(G)_{\rm Quillen}$ the corresponding reduced groups. Thus $\bar{H}_0(G)_{\rm Quillen} = 0$ and $\bar{H}_i(G)_{\rm Quillen} = H_i(G)_{\rm Quillen}$ for $i > 0$. Then both $H_*(G)_{\rm Top}$ and $H_*(G)_{\rm Quillen}$ are functors $\mathcal{C}^n\mathcal{G} \to \mathsf{Ab}$ to the category of abelian groups. When $n = 0$ we have functors $H_*(-)_{\rm Top}, H_*(-)_{\rm Quillen}\colon \mathcal{G} \to \mathsf{Ab}$ and it is well-known that

$$H_*(G)_{\rm Top} \cong H_*(G)_{\rm Quillen}$$

in this case. We denote both of these homology functors by $H_*(G)$.

Let us now consider $n = 1$. A cat^1-group G is equivalent to a crossed module $\lambda_1\colon \mathcal{E}G_{\{1\}} \to \mathcal{E}G_\emptyset$ which for simplicity we denote by $\lambda\colon M \to P$. To the group P we can associate the crossed module $0 \to P$. The inclusion morphism of crossed modules $(0 \xrightarrow{0} P) \longrightarrow (M \xrightarrow{\lambda} P)$ induces a map of simplicial sets

$$f_G\colon \mathcal{B}\begin{pmatrix} 0 \\ \downarrow \\ P \end{pmatrix} \to \mathcal{B}\begin{pmatrix} M \\ \downarrow^{\lambda} \\ P \end{pmatrix}$$

We denote by Cof(G) the homotopy cofibre of f_G. The following theorem, modulo some notation, was proved in [8]. (A more general version of the result for homology and cohomology with arbitrary coefficient module is contained in [10].)

Theorem 11 *For any cat^1-group G there is an isomorphism*

$$\bar{H}_i(G)_{\text{Quillen}} \cong H_{i+1}(|\text{Cof}(G)|), \ \ (i \geq 0)$$

and consequently an exact sequence

$$\cdots \to H_{i+1}(P) \to H_{i+1}(G)_{\text{Top}} \to H_i(G)_{\text{Quillen}} \to H_i(P) \to \cdots \quad (i \geq 1).$$

We wish to explain how this result generalises to cat^n-groups, $n \geq 1$. To pave the way we recall the proof for the case $n = 1$.

Proof. Let $Q \to G$ be a cofibrant simplicial resolution of G, that is a fibration in $\mathcal{S}(\mathcal{CG})$ which is also a weak equivalence and where Q is cofibrant. Then $\mathcal{B}Q \to \mathcal{B}G$ is a weak equivalence in $\mathcal{SS}$. Moreover, it is readily checked that $\mathcal{B}(\mathcal{E}Q_\emptyset) \to \mathcal{B}(\mathcal{E}G_\emptyset)$ is also a weak equivalence. The map f_Q and cofibre $\text{Cof}(Q)$ are defined analogously to f_G and $\text{Cof}(G)$. The homology exact sequences associated to the cofibrations f_G, f_Q show that $\text{Cof}(Q) \to \text{Cof}(G)$ induces an isomorphism in homology. Since $\text{Cof}(Q)$ and $\text{Cof}(G)$ are both 1-connected it follows that $\text{Cof}(Q) \to \text{Cof}(G)$ is a weak equivalence.

The simplicial set $\text{Cof}(Q)$ is obtained as the diagonal of a bisimplicial set X with $X_{*p} = \text{Cof}(Q_p)$, where Q_p is the p-th component of Q. The homology spectral sequence for the bisimplicial set X has the form

$$E^1_{pq} = H_q(\text{Cof}(Q_p)) \Rightarrow H_{p+q}(\text{Cof}(G))\,.$$

Now $\text{Cof}(Q_p)$ is the cofibre of the map $\mathcal{B}(P_p) \longrightarrow \mathcal{B}(M_p \to P_p)$ where $M_p \to P_p$ is the crossed module equivalent to Q_p. Since Q_p is a projective cat^1-group it follows that $M_p \to P_p$ is a projective crossed module. It is readily seen that P_p must be a free group. Part (iv) of Lemma 8 implies that both classifying spaces here have free fundamental group and trivial higher homotopy groups. So $\text{Cof}(Q_p)$ is simply connected and the homology exact sequence of a cofibration implies that $H_i(\text{Cof}(Q_p)) = 0$ for $i > 2$ and

$$H_2(\text{Cof}(Q_p)) \cong \mathsf{Ker}((P_p)_{ab} \to (P_p/M_p)_{ab}).$$

Lemma 8 implies that P_p/M_p is free. Hence $H_2(P_p/M_p) = 0$ and

$$0 \to M_p \to P_p \to P_p/M_p \to 0$$

is a split short exact sequence. It follows that

$$\mathsf{Ker}((P_p)_{ab} \to (P_p/M_p)_{ab}) \cong M_p/[M_p, P_p].$$

Thus

$$H_2(\text{Cof}(Q_p)) \cong M_p/[M_p, P_p].$$

Hence

$$E^1_{pq} = 0 \text{ if } q \neq 0 \text{ or } 2, \quad E^1_{p0} = \mathbf{Z}, \quad \text{and } E^1_{p2} = M_p/[M_p, P_p].$$

Thus E^1_{p0} is a constant simplicial abelian group. Hence $E^2_{p0} = 0$ for $p > 0$. Therefore the spectral sequence degenerates and gives the isomorphism

$$H_{i+2}(\text{Cof}(G)) \cong \pi_i(M_*/[M_*, P_*]), \quad i \geq 0.$$

Lemma 10 implies

$$\frac{M_*}{[M_*, P_*]} \cong \sigma\left(\frac{Q_*}{[Q_*, Q_*]}\right)$$

and so

$$\pi_i(M_*/[M_*, P_*]) \cong H_{i+1}(G)_{\text{Quillen}}.$$

□

Corollary 12 *Let $M \to P$ denote the crossed module associated to the cat^1-group G. If P is a free group then there are natural isomorphisms*

$$H_{i+1}(G)_{\text{Top}} \cong H_i(G)_{\text{Quillen}} \quad (i \geq 2),$$

$$H_2(G)_{\text{Top}} \cong \mathsf{Ker}(M/[M, P] \to P/[P, P]).$$

The description of $H_2(G)_{\text{Top}}$ given in the corollary can be viewed as a generalization of Hopf's formula for the second integral homology of a group K. To see this, note that if $\pi_1 G = K, \pi_2 G = 0$ in the corollary, then M is a normal subgroup of the free group P with $K \cong P/M$, and $H_2(G)_{\text{Top}} \cong H_2(K, \mathbf{Z})$. We thus recover Hopf's formula $H_2(K, \mathbf{Z}) \cong M \cap [P, P]/[M, P]$.

Consider now $n = 2$. An arbitrary cat^2-group G is equivalent to a crossed square $\mathcal{E}G$ which, for simplicity, we denote by

$$\begin{array}{ccc} L & \to & N \\ \downarrow & & \downarrow \\ M & \to & P \end{array}.$$

By applying the classifying functor $\mathcal{B}\colon \mathcal{X}^2\mathcal{G} \to \mathcal{SS}$ to a diagram of crossed squares we obtain the following diagram of simplicial sets:

$$\begin{array}{ccc} \mathcal{B}\begin{pmatrix} 0 & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & P \end{pmatrix} & \overset{f_G^1}{\to} & \mathcal{B}\begin{pmatrix} 0 & \to & N \\ \downarrow & & \downarrow \\ 0 & \to & P \end{pmatrix} \\ \downarrow g_G^1 & & \downarrow g_G^2 \\ \mathcal{B}\begin{pmatrix} 0 & \to & 0 \\ \downarrow & & \downarrow \\ M & \to & P \end{pmatrix} & \overset{f_G^2}{\to} & \mathcal{B}\begin{pmatrix} L & \to & N \\ \downarrow & & \downarrow \\ M & \to & P \end{pmatrix} \end{array}$$

There is a natural map

$$g_G\colon \text{cofibre}(f_G^1) \to \text{cofibre}(f_G^2)$$

from the homotopy cofibre of f_G^1 to the homotopy cofibre of f_G^2. We denote by $\text{Cof}(G)$ the cofibre of this map g_G.

Lemma 13 *Let G be a cat^2-group equivalent to the crossed square*

$$\begin{array}{ccc} L & \to & N \\ \downarrow & & \downarrow \\ M & \to & P. \end{array}$$

If G is a projective object in the category $\mathcal{C}^2\mathcal{G}$, then $|\text{Cof}(G)|$ is homotopy equivalent to a wedge of 3-spheres and

$$H_3(|\text{Cof}(G)|) \cong \frac{L}{[M, N][L, P]}$$

Proof. By Lemma 9 both $N \to P$ and $M \to P$ are projective objects in the category of crossed modules and hence are injections. By Proposition 1 of [8] $\mathrm{Cof}(f_G^1)$ is a wedge of 2-spheres and

$$H_2(|\mathrm{Cof}(f_G^1)|) \cong \frac{N}{[P,N]}.$$

The map f_G^2 yields the following epimorphism of free groups after applying the functor π_1

$$P/M \to P/MN.$$

Since $|\mathrm{Cof}(f_G^2)|$ is connected it follows that $|\mathrm{Cof}(f_G^2)|$ is 1-connected. On the other hand both spaces $B(M \to P)$ and $B(G)$ are homotopy equivalent to wedges of 1-spheres thanks to Lemma 8 and Lemma 9. Thus it follows from the homology exact sequence that $|\mathrm{Cof}(f_G^2)|$ is homotopy equivalent to the wedge of 2-spheres and the sequence

$$0 \to H_2(|\mathrm{Cof}(f_G^2)|) \to (P/M)_{ab} \to (P/MN)_{ab} \to 0$$

is exact. Since G is projective we have $L = M \cap N$ because $\pi_2(\mathcal{N}(G)) = 0$. Thus

$$H_2(|\mathrm{Cof}(f_G^2)|) \cong \frac{N}{[P,N] \cap M}$$

The map $\mathrm{Cof}(f_G^1) \to \mathrm{Cof}(f_G^2)$ yields the following epimorphism of groups by applying the functor H_2:

$$\frac{N}{[P,N]} \to \frac{N}{[P,N] \cap M}.$$

Hence the homology exact sequence shows that $|\mathrm{Cof}(G)|$ is a wedge of 3-spheres and that $H_3(\mathrm{Cof}(Q)) = L/[N,P] \cap M$. Since P/N is a free group the Hopf formula for $H_2(P/N)$ implies that $[N,P] = N \cap [P,P]$ and hence that $H_3(\mathrm{Cof}(Q)) = L/L \cap [P,P]$. The Hopf type formula for the third integral homology of a group [1] states that

$$H_3(P/MN) \cong \frac{L \cap [P,P]}{[M,N][L,P]}.$$

Since P/MN is a free group it follows that

$$H_3(\mathrm{Cof}(Q)) \cong \frac{L}{[M,N][L,P]}.$$

□

The following is the main result.

Theorem 14 *For any cat^2-group G there is an isomorphism*

$$\bar{H}_i(G)_{\text{Quillen}} \cong H_{i+2}(|\mathrm{Cof}(G)|), \ (i \geq 0).$$

Proof. Let $Q \to G$ be a cofibrant simplicial resolution of G. The cofibre $\mathrm{Cof}(Q)$ is defined analogously to $\mathrm{Cof}(G)$. It is readily checked that there are weak equivalences

$$\mathcal{B}(Q) \to \mathcal{B}(G),$$

$$\mathcal{B}(\mathcal{E}Q_{\{1\}} \to \mathcal{E}Q_{\emptyset}) \to \mathcal{B}(\mathcal{E}G_{\{1\}} \to \mathcal{E}G_{\emptyset}),$$

$$\mathcal{B}(\mathcal{E}Q_{\{2\}} \to \mathcal{E}Q_{\emptyset}) \to \mathcal{B}(\mathcal{E}G_{\{2\}} \to \mathcal{E}G_{\emptyset}),$$
$$\mathcal{B}(\mathcal{B}Q_{\emptyset}) \to \mathcal{B}(\mathcal{G}_{\emptyset}).$$

The homology exact sequences associated to the cofibrations

$$\mathcal{B}(\mathcal{E}G_{\{1\}} \to \mathcal{E}G_{\emptyset}) \to \mathcal{B}(G) \to \mathrm{cofibre}(f_G^2)$$
$$\mathcal{B}(\mathcal{E}Q_{\{1\}} \to \mathcal{E}Q_{\emptyset}) \to \mathcal{B}(Q) \to \mathrm{cofibre}(f_Q^2)$$

show that the map $\mathrm{cofibre}(f_Q^2) \to \mathrm{cofibre}(f_G^2)$ is a homology equivalence and hence a weak equivalence. Similarly the map $\mathrm{cofibre}(f_Q^1) \to \mathrm{cofibre}(f_G^1)$ is a weak equivalence. Hence there is a weak equivalence

$$\mathrm{Cof}(Q) \overset{\simeq}{\to} \mathrm{Cof}(G)$$

The simplicial set $\mathrm{Cof}(Q)$ is obtained as the diagonal of a bisimplicial set X with $X_{*p} = \mathrm{Cof}(Q_p)$, where Q_p is a projective cat^2-group. The homology spectral sequence for the bisimplicial set X has the form $E^1_{pq} = H_q(\mathrm{Cof}(Q_p)) \Rightarrow H_{p+q}(\mathrm{Cof}(G))$.

Now Q_p is equivalent to a projective crossed square

$$\begin{array}{ccc} L_p & \to & N_p \\ \downarrow & & \downarrow \\ M_p & \to & P_p \end{array}$$

According to Lemma 13 we have

$$E^1_{pq} = 0 \text{ if } q \neq 0 \text{ or } 3, \quad E^1_{p0} = \mathbf{Z}, \quad \text{and } E^1_{p3} = \frac{L_p}{[M_p, N_p][L_p, P_p]}.$$

So $E^2_{p0} = 0$ for $p > 0$ and the spectral sequence yields the isomorphism

$$H_{i+3}(\mathrm{Cof}(G)) \cong \pi_i \left(\frac{L_p}{[M_p, N_p][L_p, P_p]} \right), \quad i \geq 0.$$

Lemma 10 implies

$$\frac{L_p}{[M_p, N_p][L_p, P_p]} \cong \sigma \left(\frac{Q_*}{[Q_*, Q_*]} \right)$$

and so

$$\pi_i \left(\frac{L_p}{[M_p, N_p][L_p, P_p]} \right) \cong H_{i+1}(G)_{\mathrm{Quillen}}.$$

□

Corollary 15 *In the crossed square associated to a cat^2-group G suppose that the group P is free and the crossed modules $M \to P$, $N \to P$ are projective in $\mathcal{X}G$. Then*

$$H_{i+2}(G)_{\mathrm{Top}} \cong H_i(G)_{\mathrm{Quillen}} \quad (i \geq 2),$$
$$H_3(G)_{\mathrm{Top}} \cong \mathsf{Ker} \left(\frac{L}{[M,N][L,P]} \to \frac{P}{[P,P]} \right).$$

Proof. The isomorphism follows from the homology exact sequences arising from the various cofibration sequences involved in the construction of $\mathrm{Cof}(G)$. □

The description of $H_3(G)_{\mathrm{Top}}$ in the corollary can be viewed as a generalization of the Hopf-type formula for the third integral homology of a group given in [1]. Interestingly, the formula in [1] plays a key role in the proof of this generalisation.

We leave as an exercise for the reader the formulation and proof of Theorem 14 and Corollary 15 for the case $n \geq 3$.

References

[1] R. Brown and G. Ellis, 'Hopf formulae for the higher homology of a group', *Bull. London Math. Soc.* vol. 20 (1988) 124-128.

[2] P. Carrasco, A.M. Cegarra and A. R.-Grandjeán, 'Cohomology of crossed modules', to appear in *J. Pure Applied Algebra.*

[3] R. Brown and C. Spencer, 'Double groupoids and crossed modules', *Cahiers Topologie Géom. Différentielle* 17, no. 4 (1976) 343–362. Erratum, *Cahiers Topologie Géom. Diffrentielle* 18, no. 4 (1977) 430.

[4] T. Datuashvili and T. Pirashvili, 'On (co)homology of 2-types and crossed modules', *J. Algebra.* vol. 244 (2001) 352-365.

[5] G. J. Ellis, 'Homology of 2-types', J. London Math. Soc. 46 (1992) 1-27.

[6] G. Ellis and R.J. Steiner, 'Higher-dimensional crossed modules and the homotopy groups of $(n+1)$-ads', *J. Pure Applied Algebra* vol. 46 (1987) 117-136.

[7] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York 1967 x+168 pp.

[8] A. R.-Grandjeán, M. Ladra and T. Pirashvili, 'CCG-homology of crossed modules via classifying spaces', *J. Algebra* vol. 229 (2000) 660-665.

[9] J.-L. Loday, 'Spaces with finitely many homotopy groups', *J. Pure Applied Algebra* vol. 24 (1982) 179-202.

[10] S. Paoli, *Developments in the (co)homology of crossed modules*, PhD thesis, University of Warwick, in preparation.

[11] T. Porter, 'N-types of simplicial groups and crossed N-cubes', *Topology* vol. 32 (1993) 5-24.

[12] D. Quillen, Homotopical Algebra,*Springer Lecture Notes in Math.* vol. 43 (1967).

[13] D. Quillen, 'Spectral sequences of a double semi-simplicial group', *Topology* vol. 5 (1966) 155-157.

[14] D. Quillen, 'On the (co)-homology of commutative rings', *Proc. Symp. Pure Math.* vol. 17 (1970). Providence.