

A Magnus-Witt type isomorphism for non-free groups

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Dedicated to Professor Hvedri Inassaridze

Abstract

We use the theory of nonabelian derived functors to prove that certain Baer invariants of a group G are torsion when G has torsion second integral homology. We use this result to show that if such a group has torsion-free abelianisation then the Lie algebra formed from the quotients of the lower central series of G is isomorphic to the free Lie algebra on G_{ab} . We end the paper with some related remarks about precrossed modules and partial Lie algebras.

1 Introduction

The lower central series of a group G is defined by setting $\gamma_1 G = G$ and $\gamma_{n+1} G = [\gamma_n G, G]$ for $n \geq 1$. The quotients of this series form a Lie algebra (over \mathbf{Z})

$$\Gamma G = \bigoplus_{n \geq 1} \gamma_n G / \gamma_{n+1} G$$

where the Lie bracket is induced by the commutator maps $\gamma_m G \times \gamma_n G \rightarrow \gamma_{m+n} G$, $(x, y) \mapsto [x, y] = xyx^{-1}y^{-1}$.

Also associated to the group G is the free Lie algebra $L(G_{ab})$ on the abelian group $G_{ab} = G/\gamma_2 G$. This is defined, up to isomorphism, by the following universal property: there is an additive homomorphism $\iota: G_{ab} \rightarrow L(G_{ab})$ such that, for any other Lie algebra M and additive homomorphism $\alpha: G_{ab} \rightarrow M$, there exists a unique Lie homomorphism $\bar{\alpha}: L(G_{ab}) \rightarrow M$ making the triangle

$$\begin{array}{ccc} G_{ab} & \xrightarrow{\iota} & L(G_{ab}) \\ & \searrow \alpha & \downarrow \bar{\alpha} \\ & & M \end{array}$$

commute. The existence of $L(G_{ab})$ follows from general considerations, and the universal property implies that ι must be injective.

The canonical inclusion $\phi: G_{ab} \rightarrow \Gamma G$ induces a (surjective) homomorphism $\bar{\phi}: L(G_{ab}) \rightarrow \Gamma G$. A result of Magnus [10] and Witt [13] asserts that $\bar{\phi}$ is an isomorphism when G is a free group. In this paper we shall prove the following.

Theorem 1 *If the second integral homology $H_2(G, \mathbf{Z})$ is a torsion group then $\ker \bar{\phi}$ lies in the torsion subgroup of the additive group of $L(G_{ab})$. In particular $\bar{\phi}: L(G_{ab}) \rightarrow \Gamma G$ is an isomorphism under the additional assumption that G_{ab} is torsion free.*

A variant of this theorem was proved by Strebel [12]. His proof involves the Poincaré-Birkhoff-Witt theorem and techniques from classical homological algebra. Although Theorem 1 could be deduced from Strebel's work, we shall give an alternative proof involving the theory of nonabelian derived functors developed by Hvedri Inassaride [6] and others. We shall derive the theorem from the following result on the Baer invariants

$$M^{(n)}(G) = \frac{R \cap \gamma_{n+1}F}{\gamma_{n+1}(R, F)}, \quad n \geq 1$$

which are defined in terms of a free presentation $G \cong F/R$. Here F is a free group with normal subgroup R , $\gamma_1(R, F) = R$ and $\gamma_{n+1}(R, F) = [\gamma_n(R, F), F]$ for $n \geq 1$. The invariants $M^{(n)}(G)$ are abelian groups and are independent of the choice of free presentation (*cf.* [9]). Note that Hopf's formula states $M^{(1)}(G) \cong H_2(G, \mathbf{Z})$.

Theorem 2 *If $M^{(1)}(G)$ is a torsion group then $M^{(n)}(G)$ is a torsion group for all $n \geq 1$.*

We end the paper with some related remarks about the partial Lie algebra formed from the lower Peiffer central series of a precrossed module.

2 The proof of Theorem 1

For all $n \geq 1$ a normal inclusion $N \leq G$ gives rise to a natural five term exact sequence (*cf.* [8])

$$M^{(n)}(G) \rightarrow M^{(n)}(G/N) \rightarrow N/\gamma_{n+1}(N, G) \rightarrow G/\gamma_{n+1}G \rightarrow G/N\gamma_{n+1}G \rightarrow 1 .$$

On taking $N = \gamma_2 G$ we obtain the exact sequence

$$M^{(n)}(G) \xrightarrow{\lambda} M^{(n)}(G_{ab}) \rightarrow \gamma_{n+1}G/\gamma_{n+2}G \rightarrow 0 . \quad (1)$$

It is explained in [3] that there is a natural isomorphism of abelian groups

$$L(G_{ab}) \cong \bigoplus_{n \geq 1} M^{(n)}(G_{ab}) . \quad (2)$$

When $G = F$ is a free group this isomorphism is precisely the Magnus-Witt isomorphism. The isomorphism is obtained for arbitrary G by describing $L((F/R)_{ab})$ as a quotient of $L(F_{ab})$. We can combine (1) and (2) to obtain an exact sequence

$$\bigoplus_{n \geq 1} M^{(n)}(G) \xrightarrow{\oplus \lambda} L(G_{ab}) \xrightarrow{\bar{\phi}} \Gamma G \rightarrow 1 .$$

The first assertion of Theorem 1 follows from this sequence and Theorem 2. The second assertion then follows from the following claim applied to $A = G_{ab}$.

Claim 1 *If A is a torsion-free abelian group then the additive group of the free Lie algebra $L(A)$ is torsion-free.*

To prove the claim suppose that A is a torsion-free abelian group. Let $T(A) = \bigoplus_{n \geq 1} (\otimes^n A)$ denote the tensor algebra on A . Thus $T(A)$ is an associative algebra (over \mathbf{Z}) formed by taking the direct sum of iterated tensor products of the abelian group A . Since the tensor product of torsion-free groups is torsion-free, the abelian group underlying $T(A)$ is torsion-free. The Poincaré-Birkhoff-Witt theorem for \mathbf{Z} -modules [5] implies that the free Lie algebra $L(A)$ imbeds into $T(A)$. This proves the claim.

3 The proof of Theorem 2

Recall that a free simplicial resolution of G consists of a simplicial group $F_* = \{F_m\}_{m \geq 0}$ with $\pi_0(F_*) \cong G$, $\pi_m(F_*) = 0$ for $m \geq 1$, and with the F_m free groups whose bases are preserved by the boundary and degeneracy morphisms $d_i^m: F_m \rightarrow F_{m-1}$, $s_i^m: F_m \rightarrow F_{m+1}$, $0 \leq i \leq m$. In this context the homotopy groups of F_* can be defined as $\pi_m(F_*) = \ker(\partial_m) / \text{im}(\partial_{m+1})$ where $\partial_m: \bigcap_{i=1}^m \ker d_i^m \rightarrow \bigcap_{i=1}^{m-1} \ker d_i^{m-1}$ is the restriction of $d_0^m: F_m \rightarrow F_{m-1}$. In other words $\pi_m(F_*)$ is the m th homology group of the Moore complex

$$N(F_*) : \cdots N(F_*)_m \xrightarrow{\partial_m} N(F_*)_{m-1} \rightarrow \cdots \rightarrow N(F_*)_0 \rightarrow 1$$

with $N(F_*)_m = \bigcap_{i=1}^m \ker d_i^m$.

Given a functor $T: (\text{Groups}) \rightarrow (\text{Groups})$ we define its left derived functors as

$$L_m^T(G) = \pi_m(T(F_*)), \quad m \geq 0.$$

The groups $L_m^T(G)$ are independent of the choice of free simplicial resolution. For more details see for instance [6].

We shall consider the derived functors of the functor

$$\tau_n: (\text{Groups}) \rightarrow (\text{Groups}), \quad G \mapsto G / \gamma_{n+1} G$$

for $n \geq 1$. In [4] it was observed that there are natural isomorphisms

$$\begin{aligned} L_0^{\tau_n}(G) &\cong \tau_n G = G / \gamma_{n+1} G, \\ L_1^{\tau_n}(G) &\cong M^{(n)}(G). \end{aligned}$$

Let F_* be a free simplicial resolution of G . The short exact sequence of simplicial groups

$$1 \rightarrow \gamma_{n+1}(F_*) \rightarrow F_* \rightarrow \tau_n(F_*) \rightarrow 1$$

gives rise to a long exact sequence of homotopy groups part of which is

$$\pi_1(F_*) \rightarrow \pi_1(\tau_n(F_*)) \rightarrow \pi_0(\gamma_{n+1}(F_*)) \rightarrow \pi_0(F_*) \rightarrow \pi_0(\tau_n(F_*)) \rightarrow 1.$$

Using the various isomorphisms we can rewrite this as an exact sequence

$$0 \rightarrow M^{(n)}(G) \rightarrow \pi_0(\gamma_{n+1}(F_*)) \rightarrow G \rightarrow \tau_n(G) \rightarrow 1$$

from which we obtain the natural isomorphism

$$M^{(n)}(G) \cong \pi_0(\gamma_{n+1}(F_*)) \twoheadrightarrow \gamma_{n+1} G. \quad (3)$$

The short exact sequence of simplicial groups

$$1 \rightarrow \gamma_{n+2}(F_*) \rightarrow \gamma_{n+1}(F_*) \rightarrow \gamma_{n+1}(F_*)/\gamma_{n+2}(F_*) \rightarrow 1$$

and (3) yield the exact sequence

$$\pi_1(\gamma_{n+1}(F_*)/\gamma_{n+2}(F_*)) \rightarrow M^{(n+1)}(G) \rightarrow M^{(n)}(G). \quad (4)$$

Theorem 2 follows by induction from (4) and the following.

Claim 2 *If $M^{(1)}(G)$ is a torsion group then so too is $\pi_1(\gamma_{n+1}(F_*)/\gamma_{n+2}(F_*))$ for $n \geq 1$.*

To prove the claim we introduce the functors

$$\Gamma_n: (\text{Groups}) \rightarrow (\text{Groups}), G \mapsto \gamma_n G / \gamma_{n+1} G, \quad n \geq 1.$$

Thus, as an abelian group, ΓG is equal to $\bigoplus_{n \geq 1} \Gamma_n G$. Define $L_n(G_{ab})$ to be the preimage of $\Gamma_n G$ under the canonical surjection $L(G_{ab}) \rightarrow \Gamma G$. The result of Magnus and Witt gives a natural isomorphism $L_n(F_{ab}) \cong \Gamma_n F$ for any free group F .

An abelian group is torsion if and only if it is trivial when tensored by the rationals. Tensoring the Lie algebra $L(F_{ab})$ by the rationals yields a Lie algebra $L(F_{ab}) \otimes \mathbf{Q}$ over \mathbf{Q} . Let $L(F_{ab} \otimes \mathbf{Q})$ denote the free lie algebra (over \mathbf{Q}) on the vector space $F_{ab} \otimes \mathbf{Q}$. By verifying the appropriate universal property we see that there is a natural isomorphism $L(F_{ab}) \otimes \mathbf{Q} \cong L(F_{ab} \otimes \mathbf{Q})$. Let $L_n(F_{ab} \otimes \mathbf{Q})$ denote the vector subspace of $L(F_{ab} \otimes \mathbf{Q})$ generated by the canonical image of $L_n(F_{ab})$. So $L_n(F_{ab} \otimes \mathbf{Q}) \cong L_n(F_{ab}) \otimes \mathbf{Q}$.

We have

$$\begin{aligned} \pi_1(\gamma_{n+1}(F_*)/\gamma_{n+2}(F_*)) \otimes \mathbf{Q} &= \pi_1(\Gamma_{n+1}(F_*)) \otimes \mathbf{Q} \\ &\cong \pi_1(L_{n+1}((F_*)_{ab})) \otimes \mathbf{Q} \\ &= H_1(N(L_{n+1}((F_*)_{ab}))) \otimes \mathbf{Q} \\ &\cong H_1(N(L_{n+1}((F_*)_{ab}) \otimes \mathbf{Q})) \quad (5) \\ &\cong H_1(N(L_{n+1}((F_*)_{ab}) \otimes \mathbf{Q})) \quad (6) \\ &= \pi_1(L_{n+1}((F_*)_{ab}) \otimes \mathbf{Q}) \\ &\cong \pi_1(L_{n+1}((F_*)_{ab} \otimes \mathbf{Q})) \end{aligned}$$

Isomorphism (5) follows from the Universal Coefficient Theorem for chain complexes of free abelian groups (see for instance [11]). Isomorphism (6) follows from and the Dold-Kan theorem [7] (which expresses a simplicial abelian group A_* in terms of its Moore complex $N(A_*)$) and the fact that tensor products commute with direct sums.

By the Universal Coefficient Theorem $\pi_1((F_*)_{ab} \otimes \mathbf{Q}) \cong \pi_1((F_*)_{ab}) \otimes \mathbf{Q} \cong M^{(1)}(G) \otimes \mathbf{Q}$. Suppose that $M^{(1)}$ is a torsion group. Then $M^{(1)}(G) \otimes \mathbf{Q} = 0$ and in low dimensions (≤ 2) the simplicial abelian group $(F_*)_{ab} \otimes \mathbf{Q}$ coincides with a free simplicial resolution of the vector space $G_{ab} \otimes \mathbf{Q}$. Since vector spaces are ‘free objects’ the simplicial group $(F_*)_{ab} \otimes \mathbf{Q}$ admits a contracting homotopy in low dimensions. Therefore $\pi_1(L_{n+1}((F_*)_{ab} \otimes \mathbf{Q})) = 0$. This proves the claim.

4 Peiffer commutators and partial Lie algebras

The isomorphism of Magnus and Witt has been generalised by H.J. Baues and D. Conduché [1]. In the generalisation free groups are replaced by ‘free precrossed modules’ and Lie

algebras are replaced by ‘partial Lie algebras’. We suspect that Theorem 1 might also admit a generalisation in this direction.

Recall that a *precrossed module* is a homomorphism of groups $\partial: M \rightarrow P$ with an action of P on M satisfying $\partial(p m) = p m p^{-1}$ for all $m \in M, p \in P$. An element in M of the form

$$m m' m^{-1} (\partial m m')^{-1}$$

with $m, m' \in M$ is called a *Peiffer commutator* and denoted by $\langle m, m' \rangle$. A precrossed module is called a *crossed module* if all Peiffer commutators are trivial. For any subgroup N in M let us denote by $\langle N, M \rangle$ the subgroup of M generated by those Peiffer commutators with either $m \in N$ or $m' \in M$. The lower Peiffer central series is then defined by setting $M_1 = M$ and $M_{n+1} = \langle M_n, M \rangle$ for $n \geq 1$. Each term M_n is a normal subgroup of M closed under the action of P . For $n \geq 2$ the quotient group M_n/M_{n+1} is abelian. It is natural to set $C = (M/M_2)_{ab}$ and form the direct sum

$$\Gamma M = C \oplus \bigoplus_{n \geq 2} M_n/M_{n+1}.$$

The Peiffer commutator map $M_m \times M_n \rightarrow M_{m+n}, (m, m') \mapsto \langle m, m' \rangle$ induces a bilinear mapping $[-, -]: \Gamma M \times \Gamma M \rightarrow \Gamma M$. The structure $(\Gamma M, [-, -])$ is not in general a Lie algebra. It is however the motivating example of what is called a ‘partial Lie algebra’ in [1]. The main result in [1] asserts that ΓM is a free partial Lie algebra if $\partial: M \rightarrow P$ is a free precrossed module with P a free group. The Magnus-Witt isomorphism corresponds to the case when P is the trivial group (for in this case the lower Peiffer central series of M coincides with the lower central series of M considered as a group).

Integral homology groups $H_m(M)_P$ for a precrossed module were introduced in [2] for dimensions $m = 1, 2$. Theorem 1 in [2] with the main result in [1] immediately imply the following.

Proposition 1 *Let $\partial: M \rightarrow P$ be a precrossed module such that: 1) P is a free group; 2) the induced map $\partial: M/M_2 \rightarrow P$ is a free crossed module; 3) $H_2(M)_P$ is trivial. Then ΓM is a free partial Lie algebra.*

This proposition with P equal to the trivial group is a special case of Theorem 1 (since in this case: hypothesis (2) asserts that M_{ab} is a free abelian group; $H_2(M)_1$ is the second integral homology of the group M ; and ΓM is a Lie algebra). It would be interesting to know whether Proposition 1 can be generalised in the direction of Theorem 1 when P is not trivial.

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