A BOUND ON THE SCHUR MULTIPLIER OF A PRIME-POWER GROUP

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For Bernhard Neumann on his 90th birthday.

ABSTRACT. The paper improves on an upper bound for the order of the Schur multiplier of a finite p-group given in [6]. The new bound is applied to the problem of classifying p-groups according to the size of their Schur multipliers.

In a paper [6] dedicated to B.H. Neumann's sixtieth birthday, the second author used results of [5] to show that a d-generator group G of prime-power order p^n has Schur multiplier $M(G)$ of order at most $p^{(d-1)(2n-d)/2}$. In this article we use results of the first author [3] to obtain a reduction of this bound. The reduced bound is then applied to the problem of classifying p-groups according to the orders of their Schur multipliers, at least in the case where the multipliers are large.

We begin by blending parts (i) and (ii) of Proposition 5 in [3] to produce the following proposition.

Proposition 1. *[3] Let G be a finite p-group with centre $Z(G)$ and lower central series $1 = \gamma_{c+1}G \trianglelefteq \gamma_c G \trianglelefteq \cdots \trianglelefteq \gamma_1 G = G$. Set $\overline{G} = G/Z(G)$ and consider the homomorphism*

$$\Psi\colon \overline{G}^{ab} \otimes \overline{G}^{ab} \otimes \overline{G}^{ab} \longrightarrow \frac{\gamma_2 G}{\gamma_3 G} \otimes \overline{G}^{ab},\ \ \overline{x} \otimes \overline{y} \otimes \overline{z} \mapsto [x,y]_\gamma \otimes \overline{z} + [y,z]_\gamma \otimes \overline{x} + [z,x]_\gamma \otimes \overline{y}.$$

Here $\overline{x}$ denotes the image in $\overline{G}$ of the element $x \in G$, and $[x,y]_\gamma$ denotes the image in $\gamma_2 G/\gamma_3 G$ of the commutator $[x,y] \in G$. Then

$$|M(G)||\gamma_2 G||\mathrm{image}(\Psi)| \le |M(G^{ab})||\frac{\gamma_2 G}{\gamma_3 G} \otimes \overline{G}^{ab}||\frac{\gamma_3 G}{\gamma_4 G} \otimes \overline{G}^{ab}|\cdots|\gamma_c G \otimes \overline{G}^{ab}|. \tag{1}$$

Proposition 1 leads to the following numerical bound on the order of the Schur multiplier.

Theorem 2. *Let G be a d-generator group of order p^n. Suppose that the abelianisation G^{ab} has order p^m and exponent p^e, and that the central quotient $G/Z(G)$ is a δ-generator group. Then*

$$|M(G)| \le p^{d(m-e)/2\ +\ (\delta-1)(n-m)\ -\ \max\{0,\delta-2\}}. \tag{2}$$

Since $e \ge m/d$ and $d \ge \delta$, inequality (2) implies

$$|M(G)| \le p^{(d-1)(2n-m)/2}. \tag{3}$$

Bound (3) is attained if $G = C_{p^e} \times C_{p^e} \times \cdots \times C_{p^e}$.

Proof. Recall that $M(G^{ab})$ is isomorphic to the exterior square $G^{ab} \wedge G^{ab}$ of abelian groups [2]. Suppose that $G^{ab} \cong C_{p^{n_1}} \times C_{p^{n_2}} \times \cdots \times C_{p^{n_d}}$ where $n_1 \le n_2 \le \cdots \le n_d = e$ and $n_1 + n_2 + \cdots + n_d$ = m. Then $M(G)$ has order p^a, where

$$\begin{aligned} a &= (d-1)n_1+(d-2)n_2+\cdots+n_{d-1} \\ &= d(n_1+n_2+\cdots+n_{d-1})-(n_1+2n_2+\cdots+(d-1)n_{d-1}) \\ &= d(m-e)-(n_1+2n_2+\cdots+(d-1)n_{d-1}) \\ &\le d(m-e)-\tfrac{m-e}{d-1}(1+2+\cdots+(d-1)) \\ &= d(m-e)/2. \end{aligned} \tag{4}$$

Since the tensor product is distributive with respect to direct sums, we have

$$|\frac{\gamma_2 G}{\gamma_3 G}\otimes \overline{G}^{ab}||\frac{\gamma_3 G}{\gamma_4 G}\otimes \overline{G}^{ab}|\cdots|\gamma_c G\otimes \overline{G}^{ab}| = |(\frac{\gamma_2 G}{\gamma_3 G}\oplus\cdots\oplus\gamma_c G)\otimes \overline{G}^{ab}| \le p^{\delta(n-m)}. \tag{5}$$

Suppose next that $\delta \ge 3$. Since $\gamma_2 G/\gamma_3 G$ is non-trivial, we can choose a generating set $\{\overline{x}_1,\overline{x}_2,\cdots,\overline{x}_\delta\}$ for $G/Z(G)$ such that $[x_1,x_2]_\gamma$ is a non-trivial element of $\gamma_2 G/\gamma_3 G$ and indeed is not a pth power of any element there since pth powers lie in the Frattini subgroup. We shall establish now the critical point of the proof, *viz.* that the $\delta-2$ elements

$$\Psi(\overline{x}_1\otimes\overline{x}_2\otimes\overline{x}_3),\Psi(\overline{x}_1\otimes\overline{x}_2\otimes\overline{x}_4),\cdots,\Psi(\overline{x}_1\otimes\overline{x}_2\otimes\overline{x}_\delta) \tag{6}$$

constitute $\delta-2$ linearly independent elements in the abelian group $\gamma_2 G/\gamma_3 G\otimes\overline{G}^{ab}$. Setting $A:=\gamma_2 G/\gamma_3 G$ temporarily, we see that

$$A\otimes\overline{G}^{ab}\cong(A\otimes<\overline{x}_1>)\times\cdots\times(A\otimes<\overline{x}_\delta>),$$

and that $\Psi(\overline{x}_1\otimes\overline{x}_2\otimes\overline{x}_i)$ is the only one of the $\delta-2$ elements listed in (6) above to have a non-trivial projection in $A\otimes<\overline{x}_i>$, so that these $\delta-2$ elements are indeed linearly independent and we have

$$|\text{image}(\Psi)| \ge p^{\delta-2}. \tag{7}$$

Inequality (2) is obtained by substituting inequalities (4), (5) and (7) into (1). □

The methods in [3] show that the quantity $|\text{image}(\Psi)|$ could be replaced by a (larger) product $|\text{image}(\Psi)||\text{image}(\Psi_3)|\cdots|\text{image}(\Psi_c)|$ in inequality (1), thus leading to an improvement in the bounds of Theorem 2.

On substituting the inequalities $d\le m\le n$ into (3) we obtain a well-known result of J.A. Green, namely that $|M(G)|\le p^{n(n-1)/2}$ for any group G of order p^n. In other words, for any group G of order p^n there is an integer $t\ge 0$ such that $|M(G)|=p^{n(n-1)/2-t}$. Those finite p-groups with $t=0,1$ have been classified by Berkovich [1]. The classification has been extended to $t=2$ by Zhou [7], and to $t=3$ by the first author [4]. In light of this work we make the following formal definition.

Definition. Let the *corank* of a finite p-group G be the integer $t=\text{corank}(G)$ for which $|M(G)|=p^{n(n-1)/2-t}$ with $n=\log_p|G|$.

The known classifications of finite p-groups by corank are summarised in the following table. All groups G with $\text{corank}(G)\le 3$ are listed. In the table C_{p^i} denotes the cyclic group of order p^i, D denotes the dihedral group of order 8, Q denotes the quaternion group of order 8, E_1 denotes the extraspecial group of order p^3 with odd exponent p, and E_2 denotes the extraspecial group of order p^3 with odd exponent p^2.

corank(G)	$p=2$	$p=$ odd prime
$t=0$	$(C_2)^k,\ k\geq 1$	$(C_p)^k,\ k\geq 1$
$t=1$	C_4	$C_{p^2},\ E_1$
$t=2$	$C_2\times C_4,\ D$	$C_p\times C_{p^2},\ E_1$
$t=3$	$C_8,\ C_2\times C_2\times C_4,\ Q,\ D\times C_2$	$C_{p^3},\ C_p\times C_p\times C_{p^2},\ E_2,\ E_1\times C_p\times C_p$

In [4] it is shown how the information in this table can be derived from a bound on the Schur multiplier due to Gaschütz, Neubüser and Yen [5]. Since inequality (2) is slightly sharper than the bound of Gaschütz *et. al.*, it too has ramifications for the classification of p-groups by corank. Some of these are listed in the following proposition. An interesting corollary to the proposition is that, for any given prime p and integer $t\geq 1$, there are only finitely many p-groups G with corank$(G)=t$.

Proposition 3. *Let G be a non-cyclic d-generator group of order p^n, with commutator subgroup $[G,G]$ of order p^c, and Frattini subgroup $[G,G]G^p$ of order p^a. Suppose that the abelianisation G^{ab} has exponent p^e, and that the central quotient $G/Z(G)$ is a δ-generator group. Furthermore, suppose that* corank$(G)=t$ *where $t\geq 1$. Then :*

(i) $0\leq c\leq t$.

(ii) $c\leq a\leq\sqrt{2t-c}$.

(iii) $2\leq d\leq\frac{2(t+a)-a^2-3c}{a-c}$ *whenever* $a\neq c$.

(iv) $2\leq d\leq t+2-\frac{a^2+a}{2}$ *whenever* $a=c$.

(v) $\frac{a^2-2a+(d+3)c+ad-2(t+1)}{2c-1}\leq\delta\leq d$ *whenever* $c\neq 0$.

(vi) $1\leq e\leq\frac{2t-2(d+1-\delta)c-d(a-c-1)-(a^2-a)-2\max\{0,\delta-2\}}{d}$.

(vii) $\frac{1+\sqrt{1+4t}}{2}\leq n\leq\frac{2t+a(c+e)+2(\delta-1)c-2\max\{0,\delta-2\}}{c+e+a-1}$.

Proof. Note that $a\geq c\geq 0$, $d\geq\delta\geq 0$, $e\geq m/d\geq 1$ and $d\geq 2$. On substituting $n=a+d, m=a+d-c$ into (2) we obtain

$$a^2-a\leq 2(t-(d+1-\delta)c)+d(c+1-a-e)-2\max\{0,\delta-2\}. \tag{8}$$

We derive the inequality

$$a^2-a\leq 2(t-c)-(a-c)(d-1)-2\max\{0,\delta-2\} \tag{9}$$

from (8) by substituting $d\geq\delta, e\geq m/d$. Since $a^2-a\geq 0$, inequality (9) implies (i). Since $d-1\geq 1$, inequality (9) implies $a^2\leq 2t-c$, from which we deduce (ii). We also deduce (iii) from (9). On substituting $a=c$, $e=1$, $\delta\geq 2$ into (8), we obtain

$$d+(d-\delta)(a-1)\leq t+2-\frac{a^2+a}{2}. \tag{10}$$

The inequality $\delta\geq 2$ corresponds to the fact [2] that no non-trivial cyclic group is itself a central quotient. Inequality (10) implies (iv). Inequality (9) implies (v), the condition $c\neq 0$ being used to obtain $\delta\geq 2$. Inequality (8) implies (vi) and the right-hand inequality of (vii). The left-hand inequality of (vii) follows immediately from the definition of corank. □

Corollary 4. *(i) For each prime p and integer $t \geq 0$ there exists at least one p-group with corank equal to t.*
(ii) For each prime p and integer $t \geq 1$ there are only finitely many p-groups with corank equal to t.

Proof. The formula for the Schur multiplier of a direct product [2], namely $M(G \times H) \cong M(G) \oplus M(H) \oplus (G^{ab} \otimes H^{ab})$, can be used to show that the abelian group $(C_p)^{t-1} \times C_{p^2}$ has corank equal to t for each $t \geq 1$. Any elementary abelian group has corank equal to 0. This proves part (i).

Suppose that G is a p-group with $\mathrm{corank}(G) = t \geq 1$. Proposition 3 implies that the order of G is bounded by a number, say $f(t)$, that depends only on t. There are only finitely many groups of order at most $f(t)$. This proves part (ii). □

The following modification to the definition of corank provides a single numerical parameter for measuring how far a p-group 'deviates' from being elementary abelian.

Definition The *relative corank* of a finite p-group G is the number

$$\mathrm{rcrank}(G) = \frac{\mathrm{corank}(G)}{\log_p |G|}.$$

Thus the relative corank is a rational number lying in the range

$$0 \leq \mathrm{rcrank}(G) \leq \frac{\log_p |G| - 1}{2}.$$

Proposition 3(ii) shows that groups with a small relative corank also have a relatively small Frattini subgroup. But relative corank captures more than the size of the Frattini subgroup. For example, the dihedral and quaternion groups of order eight have $\mathrm{rcrank}(D) = 2/3$ and $\mathrm{rcrank}(Q) = 1$. For certain families of groups it is fairly straightforward to compute the relative corank. For instance, letting $ES(p,k)$ denote an arbitrary extraspecial p-group of order p^{2k+1}, we have:

$\mathrm{rcrank}((C_p)^n) = 0,$

$\mathrm{rcrank}(C_{p^n}) = \frac{n-1}{2},$

$\mathrm{rcrank}((C_p)^{n-2} \times C_{p^2}) = \frac{n-1}{n},$

$\mathrm{rcrank}((C_{p^2})^{n/2}) = \frac{n}{4},$

$\mathrm{rcrank}(ES(p,k)) = 1, \quad \text{for } k \geq 2,$

$\mathrm{rcrank}(ES(p,k) \times ES(p,k)) = 2 + \frac{1}{4k+2}, \quad \text{for } k \geq 2.$

To obtain the last two calculations we have used the description of the Schur multipliers of extraspecial p-groups given in [2], together with the following simple lemma whose proof is left to the reader.

Lemma 5. *Let G and H be groups of orders p^n and p^m. Then*

$$\mathrm{rcrank}(G \times H) = \frac{n}{n+m}\mathrm{rcrank}(G) + \frac{m}{n+m}\mathrm{rcrank}(H) + \frac{nm - \log_p |G^{ab} \times H^{ab}|}{n+m}.$$

Acknowledgement. The first author was based at the Max-Planck-Institut für Mathematik, Bonn during the course of this work. He would like to thank the Institute for its very generous hospitality.

References

[1] Ya. G. Berkovich, 'On the order of the commutator subgroup and Schur multiplier of a finite p-group', *J. Algebra*, **144** (1991), 269-272.

[2] F.R. Beyl and J. Tappe, *Group extensions, representations, and the Schur multiplicator*, Lecture Notes in Math., **958** (Springer-Verlag, 1982).

[3] G. Ellis, 'A bound for the derived and Frattini subgroups of a prime-power group', *Proc. Amer. Math. Soc.*, **126** No. 9 (1998), 2513-2523.

[4] G. Ellis, 'On the Schur multiplier of p-groups', *Comm. Algebra*, to appear.

[5] W. Gaschütz, J. Neubüser and Ti. Yen, 'Über den Multiplikator von p-Gruppen', *Math. Zeitschrift*, **100** (1967), 93-96.

[6] J. Wiegold, 'Commutator subgroups of finite p-groups', *J. Australian Math. Soc.*, **10** (1969), 480-484.

[7] X. Zhou, On the order of Schur multipliers of finite p-groups, *Comm. Algebra*, **22** no. 1 (1994), 1-8.

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