Enumerating prime-power homotopy k-types

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1 Introduction

A fundamental problem in algebraic topology is the classification of homotopy types of polyhedra [2]. In this article we use classical techniques to enumerate particular families of such homotopy types. Recall [1] [2] that a *homotopy k-type* is the class of all topological spaces that are homotopy equivalent to some given connected polyhedron X whose homotopy groups $\pi_i X$ are trivial for $i \geq k+1$. We define the *order* of such a k-type to be the product $|\pi_1 X| \times |\pi_2 X| \times ... \times |\pi_k X|$ of the orders of the homotopy groups of X, and for integers $k, m \geq 1$ we denote by $\Lambda(k,m)$ the number of homotopy k-types of order m. Note that $\Lambda(1,m)$ is equal to the number of groups of a given order m. Higman [5] and Sims [6] have shown that the number of groups of prime-power order p^n is

$$\Lambda(1,p^n) = p^{\frac{2}{27}n^3 + O(n^{\frac{8}{3}})}. \tag{1}$$

Our main result is the following higher-dimensional analogue of this estimate.

Theorem 1. *For integers $k \geq 2, n \geq 1$ and p a prime, the number of homotopy k-types of order p^n is*

$$\Lambda(k,p^n) = p^{\frac{(k+1)^{k+1}}{(k+1)!(k+2)^{k+2}}n^{k+2} + O(n^{k+1})}.$$

Thus, for large n, there are roughly $p^{\frac{9}{512}n^4}$ homotopy 2-types of order p^n; there are roughly $p^{\frac{32}{9375}n^5}$ homotopy 3-types of order p^n, and so on.

In the proof of Theorem 1 we use the spectral sequence of a fibration to show that a high proportion of homotopy k-types X of order p^n have homotopy groups $\pi_i X$ that are elementary abelian in dimensions $i = 1, k$ and trivial in dimensions $i \neq 1, k$.

Throughout the article p denotes a prime, and $\binom{n}{i}$ denotes the coefficient of x^i in the polynomial $(x+1)^n$.

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2 Cohomological bounds

We need the following bound on the order of the qth cohomology $H^q(X, \mathbf{Z}_p)$ of a prime-power homotopy k-type X with trivial coefficients in the integers modulo p. (The corresponding integral homology bound is of independent interest; for $k=1$ and $q=2$ the homology bound is a much used result of J.A. Green [4] [8] on the Schur multiplier.)

Proposition 2. *Let $k, q \geq 1$ be integers, and let X be a finite homotopy k-type of order p^n. Then*

$$\log_p |H^q(X, \mathbf{Z}_p)| \leq \binom{n+q-1}{q},$$

$$\log_p |H_q(X, \mathbf{Z})| \leq \sum_{i=1}^{q} \binom{n+i-1}{i} (-1)^{q-i},$$

and these bounds are attained when $k=1$ and $\pi_1 X$ is elementary abelian.

Proof. Let $D^q(n)$ and $D_q(n)$ denote the ranks of the elementary abelian groups $H^q(E_n, \mathbf{Z}_p)$ and $H_q(E_n, \mathbf{Z})$, where E_n is the homotopy 1-type whose fundamental group is elementary abelian of order p^n. The homology Künneth formula [7] applied to the direct product $E_n = E_1 \times E_{n-1}$ yields the difference equation:

$$D_q(n+1) = D_q(1) + \sum_{i=1}^{n} D_i(n), \quad D_q(1) = 1 \text{ for odd } q, \quad D_q(1) = 0 \text{ for even } q. \tag{2}$$

The Universal Coefficient Theorem [7] implies the relationship

$$D^q(n) = D_q(n) + D_{q-1}(n). \tag{3}$$

It is readily verified that

$$D^q(n) = \binom{n+q-1}{q}, \quad D_q(n) = \sum_{i=1}^{q} \binom{n+i-1}{i} (-1)^{q-i}$$

is the unique solution to equations (2) and (3), and thus the bounds of the proposition are attained when $k = 1$ and $\pi_1 X$ is elementary abelian. It remains to show that if $k \geq 1$ and $|X| = p^n$ then $|H^q(X, \mathbf{Z}_p)| \leq |H^q(E_n, \mathbf{Z}_p)|$ and $|H_q(X, \mathbf{Z})| \leq |H_q(E_n, \mathbf{Z})|$.

To obtain the cohomological inequality suppose that $\pi_k X \neq 0$ and let N be a submodule of $\pi_k X$ such that $|N| = p$ and $G = \pi_1 X$ acts trivially on N. (For $k = 1$ recall that any non-trivial prime-power group has a central subgroup of prime order. For $k \geq 2$ note that if A is any non-trivial $\mathbf{Z}G$-module, and if the orders of G and A are both finite powers of p, then there is a submodule of A of order p with trivial G-action. This is true because the submodule $Z_G(A) = \{a \in A \ : \ g.a = a \text{ for all } g \in G\}$ contains a non-zero element, the existence of which follows from the equation $|A| = |Z_G(A)| + \sum_{a \in T} |[a]|$ where $[a] = \{a' \in A : a' = g.a \text{ for some } g \in G\}$ and T is a set of representatives for the equivalence classes $[a]$.)

We can assume that X is a CW-space. By attaching cells to X in dimensions greater than k we can construct an inclusion of homotopy k-types $X \hookrightarrow Y$ which induces isomorphisms $\pi_i X \cong \pi_i Y$ for $i < k$ and $\pi_k Y \cong \pi_k X/N$. Thus $|Y| = p^{n-1}$. (In order to justify this construction it may be helpful to consider the categorical equivalence $X \mapsto SX$ between homotopy types and simplicial groups [3]. The Eilenberg-Mac Lane space $K(N, k)$ with kth homotopy group isomorphic to N is represented by a normal simplicial subgroup $SK(N, k)$ of SX, and the inclusion $X \hookrightarrow Y$ corresponds to the quotient homomorphism $SX \twoheadrightarrow SX/SK(N, k)$ of simplicial groups.) By modifying X up to homotopy type we obtain a fibration sequence

$$F \hookrightarrow X \twoheadrightarrow Y$$

the fibre of which is an Eilenberg-Mac Lane space $F = K(N, k)$.

The spectral sequence of a fibration [7]

$$E_2^{ij} = H^i(Y, H^j(F, \mathbf{Z}_p)) \Rightarrow H^{i+j}(X, \mathbf{Z}_p)$$

yields

$$|H^q(X, \mathbf{Z}_p)| = \prod_{i+j=q} |E_\infty^{i,j}| \leq \prod_{i+j=q} |E_2^{i,j}| = \prod_{i+j=q} |H^i(Y, H^j(F, \mathbf{Z}_p))|.$$

Now $H^j(F, \mathbf{Z}_p)$ is bijective with the set $[K(\mathbf{Z}_p, k), K(\mathbf{Z}_p, j)]$ of homotopy classes of maps $K(\mathbf{Z}_p, k) \to K(\mathbf{Z}_p, j)$ between Eilenberg-Mac Lane spaces. So if $j \geq k$ then, by repeated application of the loop functor [7], we see that $H^j(F, \mathbf{Z}_p) = [K(\mathbf{Z}_p, k), K(\mathbf{Z}_p, j)] = [\Omega^{k-1} K(\mathbf{Z}_p, k), \Omega^{k-1} K(\mathbf{Z}_p, j)] = [K(\mathbf{Z}_p, 1), K(\mathbf{Z}_p, j-k+1)] = H^{j-k+1}(N, \mathbf{Z}_p) \leq \mathbf{Z}_p$. Furthermore, $H^0(F, \mathbf{Z}_p) = \mathbf{Z}_p$ and $H^j(F, \mathbf{Z}_p) = 0 \leq \mathbf{Z}_p$ if $0 < j < k$. Hence

$$|H^q(X, \mathbf{Z}_p)| \leq \prod_{i+j=q} |H^i(Y, \mathbf{Z}_p)| = |H^q(Y \times K(N, 1), \mathbf{Z}_p)|$$

where the last equality follows from the Künneth formula for cohomology.

An easy inductive argument gives $|H^q(X,\mathbf{Z}_p)| \leq |H^q(E_n,\mathbf{Z}_p)|$. The homological inequality is proved in a similar way. □

Proposition 2 generalizes to arbitrary prime-power coefficients. We need the cohomological version.

Proposition 3. *Let X be a homotopy k-type of order p^n. Let A be a finite $\mathbf{Z}\pi_1 X$-module of order p^a. Then for any integer $q \geq 1$ we have*

$$\log_p |H^q(X,A)| \leq a \times \binom{n+q-1}{q}.$$

This bound is attained if $k=1$ and both $\pi_1 X$ and A are elementary abelian groups (with $\pi_1 X$ acting trivially on A).

Proof. There exists a submodule B of A such that B has order p and trivial $\pi_1 X$-action (see above). The cohomology coefficient sequence [7]

$$\cdots \to H^q(X,B) \to H^q(X,A) \to H^q(X,A/B) \to \cdots$$

yields the inequality $|H^q(X,A)| \leq |H^q(X,B)| \times |H^q(X,A/B)|$. Repeated application of this argument gives $|H^q(X,A)| \leq |H^q(X,B)|^a$. The required inequality then follows from Proposition 2. □

3 Proof of the theorem

Let $k \geq 2$ and let X be a homotopy k-type represented by a CW-space. By attaching cells to X in dimensions greater than k, we can produce an inclusion $X \hookrightarrow \overline{X}$ with $\overline{X}$ a homotopy $(k-1)$-type; there are isomorphisms $\pi_i X \cong \pi_i \overline{X}$ for $i < k$. It is well-known [7] that the homotopy k-type X determines, and is uniquely determined by, the homotopy $(k-1)$-type $\overline{X}$, the $\mathbf{Z}\pi_1\overline{X}$-module $\pi_k X$ and a cohomology class $\kappa \in H^{k+1}(\overline{X},\pi_k X)$. The class κ is said to be a *Postnikov invariant* of X. Our estimate for $\Lambda(k,p^n)$ is obtained by estimating the number of possibilities for $\overline{X}$, $\pi_k X$ and κ.

It is convenient to work with logarithms to the base p. We thus fix the prime p once and for all, and define

$$\lambda(k,n) = \log_p(\Lambda(k,p^n)),$$

$$\alpha(i,j) = \max_{\overline{X}=p^i}\{\text{number of } \mathbf{Z}\pi_1\overline{X}-\text{modules of order } p^j\},$$

where in the last definition $\overline{X}$ ranges over all homotopy $(k-1)$-types of fixed order p^i. We also define

$$\kappa^{k+1}(i,j) = \max_{\overline{X}=p^i, |A|=p^j} \log_p |H^{k+1}(\overline{X},A)|$$

where A ranges over all $\mathbf{Z}\overline{X}$-modules of order p^j, and $\overline{X}$ ranges over all homotopy $(k-1)$-types of order p^i.

This notation leads to the following inequality

$$p^{\lambda(k,n)} \leq \sum_{i=0}^{n} p^{\lambda(k-1,i)+\alpha(i,n-i)+\kappa^{k+1}(i,n-i)}$$

from which we derive

$$\lambda(k,n) \leq \log_p(n+1) + \max_{0\leq i\leq n}\{\lambda(k-1,i)+\alpha(i,n-i)+\kappa^{k+1}(i,n-i)\}. \quad (4)$$

Lemma 4. *We have*

$$\max_{0\le i\le n}\{\lambda(k-1,i)+\alpha(i,n-i)+\kappa^{k+1}(i,n-i)\}=\frac{(k+1)^{k+1}}{(k+1)!(k+2)^{k+2}}n^{k+2}+O(n^{k+1}).$$

Proof. Since the semi-direct product of a group $G=\pi_1\overline{X}$ of order at most p^i with a $\mathbf{Z}G$-module A of order p^{n-i} is a group of order at most p^n, we have the crude inequality

$$\alpha(i,n-i)\le\lambda(1,n). \tag{5}$$

Furthermore, estimate (1) can be rewritten as

$$\lambda(1,n)=\frac{2}{27}n^3+O(n^{\frac{8}{3}}). \tag{6}$$

Proposition 3 gives us

$$\kappa^{k+1}(i,n-i)=(n-i)\times\binom{i+k}{k+1}=(n-i)(\frac{i^{k+1}}{(k+1)!}+f_k(i)) \tag{7}$$

where $f_k(i)$ is a polynomial in i of degree k. The polynomial $f_k(i)$ is independent of n. The derivative of $\kappa^{k+1}(i,n-i)$ with respect to i is

$$\kappa^{k+1}(i,n-i)'=\frac{(k+1)ni^k-(k+2)i^{k+1}}{(k+1)!}+(n-i)f_k'(i)-f_k(i).$$

When n is large with respect to k, and when $i=\frac{n(k+1)}{k+2}$, the derivative $\kappa^{k+1}(i,n-i)'$ is 'approximately' zero, $\kappa^{k+1}(i,n-i)'=0+O(n^k)$. This approximate zero corresponds to an approximate maximum of $\kappa^{k+1}(i,n-i)$. Hence

$$\max_{0\le i\le n}\kappa^{k+1}(i,n-i)=\frac{(k+1)^{k+1}}{(k+1)!(k+2)^{k+2}}n^{k+2}+O(n^{k+1}). \tag{8}$$

The lemma follows from (5), (6) and (8). □

Let us now turn to the proof of Theorem 1. Inequality (4) and Lemma 4 combine to give

$$\Lambda(k,p^n)\le p^{\frac{(k+1)^{k+1}}{(k+1)!(k+2)^{k+2}}n^{k+2}+O(n^{k+1})}.$$

But consider those homotopy k-types with $\pi_1X=E(m)$, $\pi_kX=E(n-m)$ and $\pi_iX=0$ for $i\ne 1,k$, where m is equal to the integer part of $\frac{n(k+1)}{k+2}$, and where $E(m)$ denotes the elementary abelian group of order p^m. Equation (8) implies that there are at least $p^{\frac{(k+1)^{k+1}}{(k+1)!(k+2)^{k+2}}n^{k+2}+O(n^{k+1})}$ such homotopy types. This proves the theorem.

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