COHOMOLOGICAL PERIODICITIES OF CRYSTALLOGRAPHIC GROUPS

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Abstract. We observe that an n-dimensional crystallographic group G has periodic cohomology in degrees greater than n if it contains a torsion free finite index normal subgroup $S \trianglelefteq G$ whose quotient G/S has periodic cohomology. We then consider a different type of periodicity. Namely, we provide hypotheses on a crystallographic group G that imply isomorphisms $H_i(G/\gamma_c T, \mathbb{F}) \cong H_i(G/\gamma_{c+d} T, \mathbb{F})$ for $\mathbb{F}$ the field of p elements and $\gamma_c T$ a term in the relative lower central series of the translation subgroup $T \le G$. The latter periodicity provides a means of calculating the mod-p homology of certain infinite families of finite p-groups using a finite (machine) computation.

1. Introduction

An *n-dimensional crystallographic group* $G \le \mathsf{Isom}(\mathbb{R}^n)$ is a discrete subgroup of the isometries of n-dimensional Euclidean space whose translations form a free abelian subgroup $T \le G$ of dimension n. The *translation subgroup* T is a finite index normal subgroup of G and the quotient $P = G/T$ is called the *point group*. We describe two cohomological periodicities arising in the context of crystallographic groups; both provide a means of calculating infinite families of cohomology groups from finite computations.

To describe the first periodicity we say that a $\mathbb{Z}G$-resolution R_* involving boundary homomorphisms ∂_* is *periodic* of period d *in degrees greater than* m if there is equality of modules $R_i = R_{i+d}$ and boundary homomorphisms $\partial_{i+1} = \partial_{i+1+d}$ for all $i \ge m$. When $m = 0$ we simply say that such a resolution is *periodic*.

Proposition 1. *Let G be an n-dimensional crystallographic group with a torsion free normal subgroup $S \trianglelefteq G$ of finite index whose quotient $Q = G/S$ admits a periodic free $\mathbb{Z}Q$-resolution of $\mathbb{Z}$ of period d. Then G admits a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ which is periodic of period d in degrees greater than n.*

Proposition 1 allows one to calculate the integral cohomology of certain crystallographic groups in all degrees by performing a computer computation of the integral cohomology in just the first few degrees. As an illustration consider the group $G = \mathrm{SpaceGroupBBNWZ}(2, 10)$ arising as the tenth group of dimension 2 in the Cryst [6] library of crystallographic groups available as part of the GAP [8] system for computational algebra. This group is generated by the four isometries

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y+1 \end{pmatrix}.$$

The point group in this case is cyclic of order 4 and admits a periodic resolution of period 2. Using the homological algebra package HAP [7] for GAP to compute the integral homology of G in degrees up to and including 4, one obtains from Proposition 1 with $S = T$ that

$$H_i(G, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_4, & i = 1 \\ \mathbb{Z}, & i = 2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4, & \text{odd } i \ge 3 \\ 0, & \text{even } i \ge 4 . \end{cases}$$

In order to compute, say, $H_3(G, \mathbb{Z})$ in GAP one can use the following commands.

```
gap> G:=SpaceGroupBBNWZ(2,10);;
gap> GroupHomology(G,3);
[ 2, 4, 4 ]
```

The second type of periodicity concerns the *lower central series* of T relative to G, defined by setting $\gamma_1 T = T$ and $\gamma_{c+1}T = [\gamma_c T, G]$ for $c \geq 1$. There is an action of $P = G/T$ on the free abelian group $\gamma_c T$ given by conjugation, $P \times \gamma_c T \to \gamma_c T, (gT, a) \mapsto gag^{-1}$. Thus $\gamma_c T$ is a $\mathbb{Z}P$-module from which the vector space $\gamma_c T \otimes_{\mathbb{Z}} \mathbb{F}$ inherits the structure of an $\mathbb{F}P$-module for $\mathbb{F}$ any field. We say that an abelian group is *n-generator* if it is generated by n elements and not generated by fewer than n elements. We say that an n-dimensional crystallographic group G is *centrally p-periodic* of period d if the following hold:

(i) the point group $P = G/T$ is a p-group for some prime p;
(ii) there exists an integer $k \geq 1$, called the *initial periodicity degree*, such that the quotient $T/\gamma_{c+1}T$ is an n-generator abelian p-group for $c \geq k$ and, in the case $p = 2$, the quotient $T/\gamma_{c+1}T$ is a direct product of cyclic groups of exponent > 2; and
(iii) $\gamma_c T \otimes_{\mathbb{Z}} \mathbb{F}$ and $\gamma_{c+d} T \otimes_{\mathbb{Z}} \mathbb{F}$ are isomorphic $\mathbb{F}P$-modules for the field $\mathbb{F}$ of p elements and all $c \geq k$.

For a centrally p-periodic group G the quotients $G/\gamma_c T$, $c \geq k$, are an infinite family of distinct finite p-groups. The extra requirement in (ii) for the case $p = 2$ is needed to ensure that $H^*(T/\gamma_{c+1}T, \mathbb{F})$ and $H^*(T/\gamma_{c+2}T, \mathbb{F})$ are isomorphic as rings.

As an example we again consider $G = \text{SpaceGroupBBNWZ}(2, 10)$. We have $P = G/T \cong C_4$ and T has basis consisting of translations t_1, t_2 by the standard basis vectors. For $c \geq 1$ the free abelian group $\gamma_{2c}T$ has basis

$$\{2^{c-1}(t_1 + t_2), 2^{c-1}(-t_1 + t_2)\}$$

and the free abelian group $\gamma_{2c+1}T$ has basis

$$\{2^c t_1, 2^c t_2\}.$$

From these bases one can deduce that G is centrally 2-periodic of period $d = 1$ and initial periodicity degree $k = 5$. We have $G/\gamma_2 T \cong C_4 \times C_2$ and $G/\gamma_3 T \cong (C_4 \times C_2) \rtimes C_2$ and, for $c \geq 3$,

$$G/\gamma_c T \cong (\ldots(((C_4 \times C_2) \rtimes C_2) \rtimes C_2)\ldots) \rtimes C_2$$

involving $c - 2$ semi-direct product symbols.

Of the seventeen 2-dimensional crystallographic groups precisely six are centrally p-periodic.

Proposition 2. *Let G be an n-dimensional crystallographic group that is centrally p-periodic of period d and initial periodicity degree k. Let $\mathbb{F}$ denote the field of p elements. Then the two homology groups $H_i(G/\gamma_c T, \mathbb{F})$ and $H_i(G/\gamma_{c+d}T, \mathbb{F})$ are isomorphic vector spaces for all $i \geq 0, c \geq k$.*

The preliminary preprint [9] contains a result related to Proposition 2; it uses spectral sequence arguments to establish hypotheses under which two p-groups E_1 and E_2, arising as semidirect products $E_i = A_i \rtimes Q$ with A_i an abelian normal subgroup, have isomorphic mod-p cohomology. Experimental evidence for periodic homological properties of the lower central quotients of certain crystallographic groups has been published by Dietrich, Eick, Feichtenschlager [5] in relation to a conjecture of Carlson on the cohomology of coclass families of p-groups [3]; it is this experimental work that led us to discover Proposition 2.

Recall that for a group K of order $|K| = p^n$ and nilpotency class $c = cl(K)$ one defines the *coclass* $cc(K) = n - c$. In [3] Carlson proved that for $p = 2$ and fixed integer $r \geq 1$ there are only a finite number of isomorphism types of cohomology ring $H^*(K, \mathbb{F})$ with $cc(K) = r$. There are a number of conjectures aimed at extending the result to odd primes p and understanding the result more fully for $p = 2$. These involve the *coclass graph* $\mathcal{G}(p, r)$, a directed graph with one vertex for each p-group of coclass r, and an an edge $H \to K$ if $K/\gamma(K) \cong H$, where $\gamma(K)$ is the last non-trivial term of the lower central series of K. The graph $\mathcal{G}(p, r)$ is a forest for each p and r. Moreover, each infinite tree in the forest has a unique maximal length path of edges. The maximal path is referred to as the *mainline path* of the tree. Proposition 2 is relevant in this context because for any mainline path there is a crystallographic group G such that the quotients $G/\gamma_c T$ are precisely the vertices in the path.

Proposition 2 allows one to use computer algorithms to calculate the homology of some infinite families of finite p-groups. As an illustration we continue with $G = \text{SpaceGroupBBNWZ}(2, 10)$. Using the HAP function for computing the poincaré series

$$P(G/T_c) = \sum_{k=0}^{\infty} \dim_{\mathbb{F}} H^k(G/T_c, \mathbb{F})\, x^k$$

we find

$$P(G/T) = 1/(-x+1)\,,$$
$$P(G/\gamma_2 T) = 1/(x^2 - 2x + 1)\,,$$
$$P(G/\gamma_c T) = 1/(-x^4 + 2x^3 - 2x + 1)\,, \quad c = 3, 4, 5.$$

In light of Proposition 2 and the isomorphism $H_i(G/\gamma_c T, \mathbb{F}) \cong H^i(G/\gamma_c T, \mathbb{F})$, these five computations provide the dimensions of the vector spaces $H_i(G/\gamma_c T, \mathbb{F})$ for all $c \geq 1$ and $i \geq 0$ since G is centrally 2-periodic of period 1 and initial periodicity degree $k = 5$. We remark that $cc(G/T_c) = 2$ for $c \geq 2$.

In order to compute, say, the Poincaré series for $Q = G/T_4$ in GAP one could use the commands

```
gap> G:=SpaceGroupBBNWZ(2,10);;
gap> Q:=RelativeCentralQuotientSpaceGroup(G,4);;
gap> n:=50;;PoincareSeries(Q,n);
(1)/(-x_1^4+2*x_1^3-2*x_1+1)
```

which return a series that is guaranteed correct in the first $n = 50$ terms. One can then apply theoretical completion criteria [2] to see that all terms of the series must then, in fact, be correct. Alternatively one could replace the final command by

```
gap> HilbertPoincareSeries(LHSSpectralSequenceLastSheet(Q));
(1)/(-x_1^4+2*x_1^3-2*x_1+1)
```

which incorporates a proof of completion using the Lyndon-Hochschild-Serre speactral sequence. This alternative command was implemented by Paul Smith and applies only to 2-groups since it uses Gröbner basis algorithms for commutative rings implemented in Singular [4]; the cohomology ring is only graded commutative for $p \neq 2$.

2. Proof of Proposition 1

Let G be an n-dimensional crystallographic group. Choose a vector $v \in \mathbb{R}^n$ and let v^G denote the orbit of v under the action of G. Let v^g denote the image of v under the action of $g \in G$. Let $||x||$ denote the Euclidean length of a vector $x \in \mathbb{R}^n$. The region

$$D_G(v) = \{x \in \mathbb{R}^n \ : \ ||x - v|| \leq ||x - v^g|| \text{ for all } g \in G\}$$

is a convex polytope with finitely many faces. The images of $D_G(v)$ under the action of G tessellate $\mathbb{R}^n$ and thus give rise to a CW-structure on $\mathbb{R}^n$. Let X denote this n-dimensional CW-space.

The cellular chain complex C_*X is a $\mathbb{Z}G$-resolution of $\mathbb{Z}$. We have $C_iX = 0$ for $i > n$. In general this resolution is not free. For $0 \leq i \leq n$ the $\mathbb{Z}G$-module C_iX is finitely generated with each generator $e \in C_iX$ having a (possibly trivial) finite stabilizer subgroup $G_e \leq G$. Note that C_iX is also a finitely generated free $\mathbb{Z}S$-module for any torsion free subgroup $S \leq G$ of finite index.

Let R^Q_* be a free $\mathbb{Z}Q$-resolution of $\mathbb{Z}$ where $Q = G/S$ is the finite quotient. Then R^Q_* is also a non-free $\mathbb{Z}G$-resolution where G acts via the quotient homomorphism $\phi\colon G \twoheadrightarrow Q$. Let $C_*X \otimes_{\mathbb{Z}} R^Q_*$ denote the tensor product of chain complexes. There is a diagonal action of G on this tensor product defined by

$$g \cdot (e \otimes f) = ge \otimes \phi(g) f$$

for $g \in G$, $e \in C_iX$, $f \in R^Q_j$, and $e \otimes f$ an element of degree $i+j$ in the tensor product. Since S acts freely on C_*X and $Q = G/S$ acts freely on R^Q_* it follows that G acts freely on the tensor product. Hence $C_*X \otimes_{\mathbb{Z}} R^Q_*$ is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

If R^Q_* is periodic of period d then, by construction, $C_*X \otimes_{\mathbb{Z}} R^Q_*$ is periodic of period d in degrees greater than n. This proves Proposition 1.

3. Proof of Proposition 2

Let G be an n-dimensional crystallographic group with translation subgroup T. Each element $g \in G$ is an affine transformation $g\colon \mathbb{R}^n \to \mathbb{R}^n, v \mapsto A_g v + B_g$ where A_g is an $n \times n$ orhtogonal matrix and $B_g \in \mathbb{R}^n$. For each $g \in G$ we define the translation $\tau(g)\colon \mathbb{R}^n \to \mathbb{R}^n, v \mapsto v + B_g$ and linear isometry $\alpha(g)\colon \mathbb{R}^n \to \mathbb{R}^n, v \mapsto A_g v$. The latter provides a surjective homomorphism $\alpha\colon G \twoheadrightarrow P_W$ onto the finite group of linear isometries $P_W = \{\alpha(g) : g \in G\}$. We have an isomorphism $P \cong P_W, gT \mapsto \alpha(g)$ which we use to identify $P = P_W$.

The group G acts on T by congugation, $(g,t) \mapsto {}^g t = gtg^{-1}$, and since T is abelian this induces an action of the point group $P = G/T$ on T. The action of $g \in G$ on $t \in T$ satisfies ${}^g t = \alpha(g) t \alpha(g^{-1})$.

Suppose now that G is centrally p-periodic with period d and initial periodicity degree $k \geq 1$. For $c \geq k$ consider the finite p-groups $A_c = T/\gamma_c T$, $G_c = G/\gamma_c T$. Let $R^{A_c}_*$ denote the minimal free $\mathbb{F}A_c$-resolution of the field $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$. This resolution is unique up to equivariant chain isomorphism. Let R^P_* be some free $\mathbb{F}P$-resolution of $\mathbb{F}$ for $P = G/T$.

Each element $gT \in P$ determines an automorphism $\iota_g\colon A_c \to A_c, t\,\gamma_c T \mapsto gtg^{-1}\,\gamma_c T$. This automorphism in turn induces a chain map $\phi^g_*\colon R^{A_c}_* \to R^{A_c}_*$. The chain map ϕ^g_* induces the identity map on $H_0(R^{A_c}_*) \cong \mathbb{F}$ and is *ι_g-equivariant* in the sense that $\phi^g_i(a \cdot e) = \iota_g(a) \cdot \phi^g_i(e)$ for $a \in A_c$, $e \in R^{A_c}_i$.

Lemma 3. *The chain maps ϕ^g_* can be constructed such that $\phi^g_* \circ \phi^{g'}_* = \phi^{gg'}_*$ for all $g, g' \in G$.*

Proof. We can construct $\mathbb{F}$-linear homomorphisms $h_i\colon R^{A_c}_i \to R^{A_c}_{i+1}$ such that $\partial_{i+1} h_i + h_{i-1}\partial_i = 1$ for $i \geq 0$ and $h_{-1} = 0$. Such a collection of homomorphisms is called a *contracting chain homotopy*. For each $g \in G$ we then construct the ι_g-equivariant chain map ϕ^g_* inductively by letting $\phi^g_0\colon R^{A_c}_0 = \mathbb{F}A_c \to R^{A_c}_0 = \mathbb{F}A_c$ send the unique free generator e^0 to $\phi^g_0(e^0) = e^0$ and defining $\phi^g_i\colon R^{A_c}_i \to R^{A_c}_i$, $i > 0$, on free generators e^i_j by $\phi^g_i(e^i_j) = h_{i-1}(\phi^g_{i-1}(\partial_i(e^i_j)))$. This construction satisfies $\phi^g_* \circ \phi^{g'}_* = \phi^{gg'}_*$ for all $g, g' \in G$. □

The following proposition is a slight adaption of Proposition 1.2 in [1]. The version in [1] concerns resolutions for semi-direct products. We need a version that can be applied to crystallographic groups. We let $\overline{g} \in G_c$ be the coset represented by $g \in G$.

Proposition 4. *Let ϕ^g_* be constructed as in Lemma 3 for $g \in G$. Then the tensor product of chain complexes $R^{A_c}_* \otimes_{\mathbb{F}} R^P_*$ over $\mathbb{F}$ admits an action of $\overline{g} \in G_c$, given by*

$$\overline{g} \cdot (e \otimes f) = \overline{\tau(g)} \cdot \phi^g_{i+j}(e) \otimes \alpha(g) \cdot f \tag{1}$$

for $e \in R^{A_c}_i$, $f \in R^P_j$. With this action $R^{A_c}_* \otimes_{\mathbb{F}} R^P_*$ is a free $\mathbb{F}G_c$-resolution of $\mathbb{F}$.

Proof. It is routine to check that (1) is a well-defined action of G_c. The action is free since A_c acts freely on $R^{A_c}_*$ and $P = G_c/A_c$ acts freely on R^P_*. Standard properties of the tensor product of chain complexes imply $H_0(R^{A_c}_* \otimes_{\mathbb{F}} R^P_*) \cong \mathbb{F}$ and $H_i(R^{A_c}_* \otimes_{\mathbb{F}} R^P_*) = 0$ for $i \geq 1$. □

From Proposition 4 we have

$$H_i(G_c, \mathbb{F}) = H_i(\ (R^{A_c}_* \otimes_{\mathbb{F}} R^P_*) \otimes_{\mathbb{F}G_c} \mathbb{F}\ ) \tag{2}$$

for $i \geq 0$.

Note that

$$(R^{A_c}_* \otimes_{\mathbb{F}} R^P_*) \otimes_{\mathbb{F}G_c} \mathbb{F} \cong ((R^{A_c}_* \otimes_{\mathbb{F}} R^P_*) \otimes_{\mathbb{F}A_c} \mathbb{F}) \otimes_{\mathbb{F}P} \mathbb{F} \tag{3}$$

and that

$$(R_*^{A_c} \otimes_{\mathbb{F}} R_*^P) \otimes_{\mathbb{F}A_c} \mathbb{F} = (R_*^{A_c} \otimes_{\mathbb{F}A_c} \mathbb{F}) \otimes_{\mathbb{F}} R_*^P \ . \tag{4}$$

Since $R_*^{A_c}$ is a minimal resolution the chain complex $C_* = R_*^{A_c} \otimes_{\mathbb{F}A_c} \mathbb{F}$ has zero boundary homomorphisms and $C_i = H_i(A_c, \mathbb{F}) \cong H^i(A_c, \mathbb{F})$ for $i \geq 0$. The action of $gT \in P$ on C_* is determined by the chain map $\phi_*^g \otimes_{\mathbb{F}A_c} \mathbb{F}\colon C_* \to C_*$; this chain map coincides with the cohomology algebra homomorphism $\phi_*^g \otimes_{\mathbb{F}A_c} \mathbb{F}\colon H^*(A_c, \mathbb{F}) \to H^*(A_c, \mathbb{F})$ induced by $\iota_g\colon A_c \to A_c$.

Since G is centrally p-periodic of period d and initial periodicity degree k we have an isomorphism of graded vector spaces $H^*(A_c, \mathbb{F}) = H^*(A_{c'}, \mathbb{F})$, and of cohomology algebra structures, for all $c, c' \geq k$. These graded vector spaces are $\mathbb{F}P$-modules and it is this module structure that determines the dimension of $H_i(G/\gamma_c T, \mathbb{F})$ and $H_i(G/\gamma_{c'} T, \mathbb{F})$ $i \geq 0$. Proposition 2 follows from (2), (3), (4) and the following two lemmas.

Lemma 5. *The $\mathbb{F}P$-module structure on the graded vector space $H^*(A_c, \mathbb{F})$, $c \geq k$, is determined by the $\mathbb{F}P$-module structure on the submodule $\oplus_{1 \leq i \leq 2} H^i(A_c, \mathbb{F})$.*

Proof. As a cohomology algebra, $H^*(A_c, \mathbb{F})$ is generated by elements in degrees ≤ 2. The point group P acts on this algebra by algebra automorphisms. □

Lemma 6. *There is an $\mathbb{F}P$-module isomorphism*

$$H^2(A_c, \mathbb{F}) \cong H^2(T, \mathbb{F}) \oplus (\gamma_c T \otimes_{\mathbb{Z}} \mathbb{F}) \ .$$

Proof. The isomorphism follows from the five-term exact cohomology sequence arising from the extension $\gamma_c T \rightarrowtail T \twoheadrightarrow A_c$ together with an analysis of the P-action. □

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