Computational homology of n-types

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Abstract

We describe an algorithm for computing the integral homology of a simplicial group and illustrate an implementation on simplicial groups arising as the nerve of a category object in the category of groups.

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1. Introduction

An *n-type* X is a connected CW-space with homotopy groups $\pi_i(X) = 0$ for all $i > n$. Up to homotopy equivalence such a space can be specified algebraically by means of a simplicial group G_* whose Moore complex is trivial in degrees greater than or equal to n. (All relevant simplicial definitions are recalled below.) More precisely, by treating each group G_i as a category with one object and constructing the nerve $N(G_i)_*$ one obtains a bisimplicial set $N(G_*)_*$. The diagonal $\Delta(G_*) = \{N(G_i)_i\}_{i\geq 0}$ is a simplicial set whose geometric realization is a CW-space $B(G_*)$. The condition on the Moore complex of G_* is sufficient to ensure that $B(G_*)$ is an n-type. The functor B induces an equivalence of categories

$$Ho(\text{Simplicial groups with Moore complex trivial in degrees } \geq n) \xrightarrow{\simeq} Ho(n\text{−types})$$

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where $Ho(C)$ denotes the category obtained from a category C by localizing with respect to those maps in C, termed *quasi-isomorphisms*, that induce isomorphisms on homotopy groups. We define the integral homology of a simplicial group G_* to be

$$H_i(G_*, \mathbb{Z}) = H_i(B(G_*), \mathbb{Z}), \quad i \geq 0.$$

On setting $n = 1$ this homology reduces to the classical homology theory of groups.

A simplicial group G_* with Moore complex of length n is uniquely determined by its terms of degree less than or equal to n, and is thus easily represented on a computer. Our goal is to describe a practical algorithm that inputs this data and returns the corresponding integral homology groups. We need to impose a computability hypothesis on the input data. Let us say that a group G is *k-computable* if we have a practical algorithm for computing the first k terms of a free $\mathbb{Z}G$-resolution R_*^G of $\mathbb{Z}$, with explicit constracting homotopy, in which each module R_j^G has finite rank. We deem G_* to be *n-computable* if its component groups G_i are $(n-i)$-computable for $i \leq n$. In this article we describe an algorithm that inputs the first n terms of an n-computable simplicial group G_* together with an integer $i < n$, and outputs the abelian invariants of the group $H_i(G_*, \mathbb{Z})$.

An implementation of the algorithm (due to the second author) is available in the HAP package (13) for the GAP computer algebra system (18). We illustrate its performance on the homology of homotopy 2-types as there is particular interest in this special case. For instance, recent work of Baues and Bleile (1) classifies, up to 2-torsion, homotopy classes of Poincaré duality complexes L in dimension 4 using *fundamental triples* comprising a homotopy 2-type X_L, a homomorphism $\omega\colon \pi_1(X_L) \to \{\pm 1\}$ and a homology class $H_4(X_L, \mathbb{Z}^\omega)$. Applications of the classification require calculations of the fourth homology of 2-types. Further interest in 2-types arises from basic constructions in group theory. Recall that a *crossed module* consists of a group homomorphism $\partial\colon M \to P$ and a group action $P \times M \to M, (p, m) \mapsto {}^p m$ satisfying

(*i*) $\partial({}^p m) = p\partial(m)p^{-1}$

(*ii*) ${}^{\partial m} m' = mm'm^{-1}$

for $m, m' \in M, p \in P$. The homotopy groups of a crossed module are defined to be $\pi_1(\partial) = M/\partial(M)$, $\pi_2(\partial) = \ker(\partial)$, $\pi_i(\partial) = 0$ for $i > 2$. Three standard examples of a crossed module are:

(1) any normal subgroup M of a group P with inclusion ∂ and action given by conjugation.

(2) the homomorphism $\partial\colon M \to Aut(M)$ from any group M to its automorphism group $P = Aut(M)$ which maps $m \in M$ to the inner automorphism $x \mapsto mxm^{-1}$.

(3) the homomorphism $\partial\colon P \otimes P \to P, x \otimes y \mapsto xyx^{-1}y^{-1}$ from the tensor square of P (4). (If P is perfect then $P \otimes P$ is just the universal central extension. If P is abelian then $P \otimes P = P \otimes_{\mathbb{Z}} P$ is the usual tensor square of $\mathbb{Z}$-modules.)

For any crossed module $\partial\colon M \to P$ one can form the semi-direct product $M \rtimes P$ and the homomorphisms $s\colon M \rtimes P \to P, (m, p) \mapsto p$ and $t\colon M \rtimes P \to P, (m, p) \mapsto (\partial m)p$. One can regard this structure as a category C^∂ with $Ob(C^\partial) = P$ the set of objects and $Arr(C^\partial) = M \rtimes P$ the set of arrows, the source and target of arrows being determined by s and t. The category composition is given by $(m, p) \circ (m', p') = (mm', (\partial m')^{-1}p)$ whenever $p = (\partial m' p')$. The category composition and group composition are compatible in the sense that C^∂ is a category object in the category of groups. Such a category object is called a *cat^1-group*. The nerve $N(C^\partial)_*$ is a simplicial group. The Moore complex of $N(C^\partial)_*$ begins with the homomorphism $\partial\colon M \to P$ and is trivial in degrees > 1.

Thus $\pi_{i+1}(\partial) = \pi_i(N(C^\partial))$ for all $i \geq 0$. See for instance (21) for more information on crossed modules and their equivalence to: (i) cat^1-groups; (ii) simplicial groups with Moore complex of length 1; (iii) homotopy 2-types.

The integral homology of a crossed module ∂ was defined in (12) to be the homology of the corresponding simplicial group

$$H_i(\partial\colon M \to P, \mathbb{Z}) = H_i(N(C^\partial)_*, \mathbb{Z}) \quad i \geq 0.$$

Theoretical aspects of this cohomology have been studied in several papers (8; 5; 22). Ana Romero (24; 25; 26) has written Lisp code for computing the integral homology of a crossed module. Her method uses the fibration sequence

$$K(\pi_2(X), 2) \to X \to K(\pi_1(x), 1)$$

associated to any homotopy 2-type X and the classical homological perturbation lemma (3) to obtain a small algebraic model for X in terms of small models for the Eilenberg Mac Lane spaces $K(\pi_2(X), 2)$ and $K(\pi_1(X), 1)$. Her code is available as a module for the Kenzo system (20) for computations in Algebraic Topology.

The present paper is motivated by the work of Romero and her collaborators. We also use (a generalized version of) the homological perturbation lemma. A difference is that, instead of constructing a small simplicial model for the total space of a fibration, we aim for a small chain complex $K(G_*)$ that is chain equivalent to the chain complex of the diagonal simplicial set $\Delta(G_*)$ of an arbitrary n-computable simplicial group G_*. A second difference is that we need no assumption about the action of the fundamental group on higher homotopy groups.

The following GAP session illustrates how our implementation can be used to compute

$$H_4(\partial\colon G \to Aut(G), \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

where G is the dihedral group of order 32. The cat^1-group C^∂ has underlying group of order 4096 and homotopy groups $\pi_1(\partial) = C_4 \times C_2$, $\pi_2(\partial) = \mathbb{Z}_2$. To reduce computations a quasi-isomorphism of cat^1-groups $C^\partial \simeq D^\partial$ is contructed. The underlying group of D^∂ is non-abelian of order 64.

```
gap> C:=AutomorphismGroupAsCatOneGroup(DihedralGroup(32));;
gap> StructureDescription(HomotopyGroup(C,1));
"C4 x C2"
gap> StructureDescription(HomotopyGroup(C,2));
"C2"
gap> D:=QuasiIsomorph(C);;
gap> N:=NerveOfCatOneGroup(D,5);;
gap> K:=ChainComplexOfSimplicialGroup(N);;
gap> Homology(K,4);
[ 2, 2, 2 ]
```

These commands took 50 seconds on a Linux 1.8 GHz laptop with 1GB ram. We are not aware of any software that could readily be used to benchmark the computation (although in principle the code of Romero (25) could be used to benchmark part of the computation since, in this example, $\pi_1(\partial)$ acts trivially on $\pi_2(\partial)$). Correctness of the computation can be partially tested by using the nerve of C^∂, rather than the nerve of D^∂, to obtain the same homology result. Analogous commands could be used to compute the homology of Eilenberg-Mac Lane spaces $K(\pi, n)$ for all $n \geq 1$. However, for $n = 1, 2, 3$ there exist far more efficient computational approaches to these spaces: the HAP package (13) provides a range of implemented algorithms for the case $n = 1$; the thesis of A. Clément (6) provides an extremely efficient implementation of a technique of Cartan for $n = 2, 3$.

2. Representing resolutions of groups

We say that a free $\mathbb{Z}G$-resolution R_*^G of $\mathbb{Z}$ is *finitely generated* if the free $\mathbb{Z}G$-module R_i^G has finite rank for each degree $i \geq 0$. The first n terms of such a resolution can be represented on a computer by storing:

- the $\mathbb{Z}G$-rank of the ith free module R_i^G $(i \leq n)$.
- the image of the kth free $\mathbb{Z}G$-generator of R_i^G under the boundary homomorphism $d_i\colon R_i^G \to R_{i-1}^G$ $(i \leq n)$.
- the image of the kth free $\mathbb{Z}$-generator of R_i^G under a contracting homotopy $h_i\colon R_i^G \to R_{i+1}^G$ $(0 \leq i \leq n-1)$.

The contracting homotopy satisfies, by definition, $h_{i-1}d_i + d_{i+1}h_i = 1$ and needs to be specified on a set of free $\mathbb{Z}$-module generators of R_i^G since it is not G-equivariant. The homotopy can be used to make algorithmic the following frequent element of choice.

> *For* $x \in \ker(d_i\colon R_i^G \to R_{i-1}^G)$ *choose an element* $\tilde{x} \in R_{i+1}^G$ *such that* $d_{i+1}(\tilde{x}) = x$.

One sets $\tilde{x} = h_i(x)$. In particular, for any group homomorphism $\phi\colon G \to G'$, the contracting homotopy on a free $\mathbb{Z}G'$-resolution $R_*^{G'}$ provides an explicit induced ϕ-equivariant chain map $\phi_*\colon R_*^G \to R_*^{G'}$.

The HAP package (13) provides implementations of practical algorithms for computing the first n terms of finitely generated free $\mathbb{Z}G$-resolutions for: finite groups (14; 15; 10), finitely generated nilpotent groups (29), crystallographic groups (13; 23), various arithmetic groups (17; 11), Coxeter groups (15) and certain Artin groups (16). In this context "practical" implies that for reasonably small values of $n \geq 0$ the rank of R_n^G is reasonably small.

The assignment $G \mapsto R_*^G$ of an arbitrary free $\mathbb{Z}G$-resolution to each group G is clearly not functorial. The assignment can be made functorial by associating the *bar resolution* B_*^G to each group. Recall that B_n^G is the free $\mathbb{Z}G$-module generated by n-tuples $[g_1|g_2|\cdots|g_n]$ $(g_i \in G)$. The boundary homomorphism is $\partial = \sum_{i=0}^n (-1)^i d_i\colon B_n^G \to B_{n-1}^G$ where

$$d_i[g_1|\cdots|g_n] = \begin{cases} g_1[g_2|\cdots|g_n] & i = 0 \\ [g_1|\cdots|g_ig_{i+1}|\cdots g_n] & 0 < i < n \\ [g_1|\cdots|g_{n-1}] & i = n \end{cases} \quad (1)$$

A contracting homotopy $h_n\colon B_n^G \to B_{n+1}^G$ is given by

$$g[g_1|\cdots|g_n] \mapsto [g|g_1|\cdots|g_n].$$

We define the *bar complex* $\overline{B}_*^G$ to be the chain complex of free abelian groups given by

$$\overline{B}_*^G = B_*^G \otimes_{\mathbb{Z}G} \mathbb{Z}$$

where $\mathbb{Z}$ has trivial G-action. The bar complex is also a functorial construction.

Note that the module B_n^G is not of finite rank when G is infinite. Even for finite groups the rank of B_n^G is large and it is typically not practical to compute the homology of G from its bar complex by directly applying the Smith Normal Form algorithm. However, for both finite and infinite groups it is straightforward to represent an arbitrary word $w \in B_n^G$ (i.e. a $\mathbb{Z}G$-linear combination of generators) on a computer, together with the formulae for $d_n(w)$ and $h_n(w)$. We deem such representations of $w, d_n(w), h_n(w)$ to constitute a computer implementation of the bar resolution B_*^G.

Given computer implementations of a finitely generated $\mathbb{Z}G$-resolution R_*^G of $\mathbb{Z}$, and of the bar resolution B_*^G, it is straightforward to compute $\mathbb{Z}G$-equivariant chain equivalences $\iota: R_*^G \to B_*^G$ and $\psi: B_*^G \to R_*^G$. These two chain maps can be represented as functions that return the images $\iota(v), \psi(w)$ of arbitrary words $v \in R_n^G, w \in B_n^G$. It is also straightforward to compute $\mathbb{Z}G$-equivariant homomorphisms $H_n: B_n^G \to B_{n+1}^G$ $(n \geq 0)$ that constitute a chain homotopy $H: \iota\psi \simeq 1$.

Definition. We call the tuple $(R_*^G, B_*^G, \iota, \psi, H)$ an *effective bar resolution.*

The notion of an effective bar resolution is a mechanism for algorithmically blending the functorial properties of the bar resolution with the computational advantages of a small finitely generated resolution. An effective bar resolution induces chain equivalences $\iota: \overline{R}_*^G \to \overline{B}_*^G$, $\psi: \overline{B}_*^G \to \overline{R}_*^G$ and chain homotopy $H: \iota\psi \simeq 1$ where $\overline{R}_*^G = R_*^G \otimes_{\mathbb{Z}G} \mathbb{Z}$.

Definition. We call the induced tuple $(\overline{R}_*^G, \overline{B}_*^G, \iota, \psi, H)$ an *effective bar complex.*

An effective bar complex is almost an example of a *chain complex with effective homology* as used in J. Rubio and F. Sergeraert's theory of *effective homology* (27; 28; 20). However, for computational reasons we drop their requirement that R_*^G be a strong deformation retract of B_*^G. We remark that Rubio and Sergeraert's theory has been applied by A. Romero (24; 25; 26) to the computation of the homology of homotopy 2-types. The application of effective bar complexes given below differs slightly from Romero's work in that: (i) it applies to homotopy n-types, and (ii) it does not explicitly use the twisted cartesian product of a fibration.

3. Computing small chain complexes for simplicial groups

Recall that a *simplicial group* G_* consists of an infinitely countable sequence of groups $G_n (n \geq 0)$ and homomorphisms $d_i^n: G_n \to G_{n-1} (n \geq 1, 0 \leq i \leq n)$ satisfying certain *simplicial identities.* The identities are succinctly expressed by saying that the simplicial group is a functor $\Delta^{op} \to (Groups)$ where Δ is the category whose objects are the sets $[n] = \{0, 1, \ldots, n\}$ $(n \geq 0)$ and whose morphisms are nondecreasing maps. The *Moore complex* MG_* of a simplicial group is the chain complex of nonabelian groups $MG_n = \cap_{1 \leq i \leq n} \ker d_i^n$, $MG_0 = G_0$ with boundary homomorphisms $d_0^n: MG_n \to MG_{n-1}$, $d_0^0 = 0$. The *homotopy groups* of a simplicial group are defined to be the homology groups of its Moore complex,

$$\pi_n(G_*) = \ker(d_0^n: MG_n \to MG_{n-1}) / \text{Image}\,(d_0^{n+1}: MG_{n+1} \to MG_n)$$

for $n \geq 0$.

If the groups G_n are abelian for $n \geq 0$ then there is a second chain complex AG_* associated to G_*. It has $AG_n = G_n$ for $n \geq 0$ with boundary homomorphism given by $\partial_n = \sum_{i=0}^{n}(-1)^i d_i^n : AG_n \to AG_{n-1}$. We shall call AG_* the *alternating chain complex*. The simplicial identities imply isomorphisms $\pi_n(G_*) \cong H_n(AG_*)$.

In general one defines a simplicial object in a category C to be a functor $\Delta^{op} \to C$. For short we use terms such as *simplicial sets* for C the category of sets and *bisimplicial sets* for C the category of simplicial sets. We use the term *bicomplex* for a chain complex in the category of chain complexes of free abelian groups.

By applying the bar complex construction to the terms in a simplicial group G_* we obtain a chain complex $\overline{B}_*^{G_*}$ of simplicial abelian groups. Taking the alternating chain complex of each simplicial abelian group $\overline{B}_i^{G_*}$ yields a bicomplex which we denote by $A\overline{B}_*^{G_*}$.

Another way to construct this bicomplex is via the functor $F\colon (Sets) \to (Abelian\ groups)$ that sends a set to the free abelian group generated by the set. By applying F to the bisimplicial set $N(G_*)_*$ we get a bisimplicial abelian group $FN(G_*)_*$. Then replacing each *horizontal* simplicial abelian group $FN(G_*)_n$, and each *vertical* simplicial abelian group $FN(G_n)_*$, by the associated alternating chain complex we obtain a bicomplex $AFN(G_*)_*$. The bicomplexes $A\overline{B}_*^{G_*}$ and $AFN(G_*)_*$ are isomorphic. The Eilenberg-Zilber Theorem (see (30)) implies that the diagonal chain complex of $AFN(G_*)_*$ is chain homotopic to the total chain complex of $AFN(G_*)_*$. We thus obtain the following alternative description of the integral homology of a simplicial group G_*.

$$H_n(G_*, \mathbb{Z}) = H_n(\mathrm{Tot}(A\overline{B}_*^{G_*})), \qquad n \geq 0. \tag{2}$$

Recall that the total chain complex has $\mathrm{Tot}(A\overline{B}_*^{G_*})_n = \oplus_{p+q=n} A\overline{B}_p^{G_q}$ and boundary homomorphism $\partial(x) = d^v(x) + \delta^h(x)$ for $x \in A\overline{B}_p^{G_q}$ where d^v, d^h are the vertical and horizontal boundaries in the bicomplex $A\overline{B}_*^{G_*}$ and $\delta^h(x) = (-1)^p d^h(x)$.

The following theorem describes our basic algorithm for computing the homology of a simplicial group. It provides easily implemented formulae for computing a small chain complex K_* from which the desired homology can be extracted by a direct application of the Smith Normal Form algorithm.

Theorem 1. Suppose that G_* is a simplicial group and that we have an effective bar complex $(\overline{R}_*^{G_p}, \overline{B}_*^{G_p}, \iota, \psi, H)$ for each $p \geq 0$. Then the total complex $\mathrm{Tot}(A\overline{B}_*^{G_*})$ is chain homotopic to a chain complex (K_*, ∂') with

$$K_n = \oplus_{p+q=n} \overline{R}_p^{G_q}$$

and with boundary homomorphism

$$\partial' = d^v + \psi\delta^h\iota + \psi\delta^h H\delta^h\iota + \psi\delta^h H\delta^h H\delta^h\iota \cdots .$$

Theorem 1 is an immediate consequence of a generalization, due to Crainic (7), of the classical homological perturbation lemma (3). This generalization is stated using the notion of a *homotopy equivalence data*

$$(L, d) \overset{\psi}{\underset{\iota}{\leftrightarrows}} (M, d),\ H \tag{3}$$

which comprises chain complexes L, M, quasi-isomorphisms ι, ψ and a homotopy $\iota\psi - 1 = dH + Hd$. A *perturbation* on (3) is a homomorphism $\delta\colon M \to M$ of degree -1 such that $(d+\delta)^2 = 0$.

Lemma 2 (Generalised homological perturbation lemma (7))**.** Suppose given a perturbation δ on a homotopy equivalence data (3) for which $(1-\delta H)^{-1}$ exists. Set $A = (1-\delta H)^{-1}\delta$. Then

$$(L, d') \overset{\psi'}{\underset{\iota'}{\leftrightarrows}} (M, d+\delta),\ H' \tag{4}$$

is a homotopy equivalence data where

$$\iota' = \iota + HA\iota,\quad \psi' = \psi + \psi AH,\quad H' = H + HAH,\quad d' = d + \psi A\iota\ \ .$$

We refer the reader to (7) for a proof of Lemma 2. Theorem 1 is obtained from the lemma by taking (M, d) to be the chain complex $(\oplus_{p+q=n} A\overline{B}_p^{G_q}, d^v)$ and taking δ to be the perturbation δ^v.

The classical perturbation lemma requires the homotopy equivalence data (3) to satisfy the extra requirements

$$\psi\iota = 1, \iota H = 0, H^2 = 0. \tag{5}$$

A version of Theorem 1 derived from the classical lemma can be found in (28). This classical version is less practical for our purposes as the extra requirement $\psi\iota$ on an effective bar complex would necessitate extra computations.

4. Nerves and quasi-isomorphisms of crossed modules

The *nerve* of a small category $s, t\colon C \to Ob(C)$ is a simplicial set NC_* with $NC_0 = Ob(C)$ and $NC_k = \{(x_1, \ldots, x_k)\colon x_i \in C, t(x_i) = s(x_{i+1})\}$ for $k \geq 1$. The first boundary homomorphisms $d_i\colon NC_1 \to NC_0$ are $d_0 = s, d_1 = t$. For $k \geq 2$ the boundary homomorphisms $d_i\colon NC_k \to NC_{k-1}$ are given by composition at the ith object: $d_0(x_1, \ldots, x_n) = (x_2, \ldots, x_k)$, $d_i(x_1, \ldots, x_k) = (x_1, \ldots, x_i x_{i+1}, \ldots, x_k)$ for $1 \leq i \leq k-1$, $d_k(x_1, \ldots, x_k) = (x_1, \ldots, x_{k-1})$.

If C is a cat^1-group then the nerve is naturally a simplicial group. In this case it is readily checked that the Moore complex of the nerve is trivial in degrees greater than 1; in degrees 0 and 1 it concides with the crossed module associated to C.

A *morphism* between two cat^1-groups $s, t\colon C \to Ob(C)$, $s', t'\colon C' \to Ob(C')$ is a group homomorphism $\phi\colon C \to C'$ satisfying $\phi s = s'\phi$, $\phi t = t'\phi$. We say that ϕ is a *quasi-isomorphism* if it induces isomorphisms on the homotopy groups $\pi_1(C) = Ob(C)/t(\ker s)$, $\pi_2(C) = \ker s \cap \ker t$ of the associated crossed module.

Quasi-isomorphic cat^1-groups have quasi-isomorphic nerves and isomorphic homology. A reasonable first step towards computing homology of a cat^1-group is thus to try to find a computationally easier quasi-isomorphic copy. In the case of a finite cat^1-group this essentially means searching for a quasi-isomorphic copy whose underlying group has lower order; we can do this as follows.

Procedure 3. (i) Given a finite cat^1-group $s, t\colon C \to Ob(C)$ we can search through the normal subgroups of $\ker s$ to find a largest normal subgroup K in $\ker s$ such that K is normal in G and $K \cap \ker t = 1$. We can then choose some generating set X_K for K and

construct the group N generated by the set $X_K \cup \{t(x) : x \in X_K\}$. Then N is a normal subgroup of C for which the quotient homomorphism $C \to C/N$ is a quasi-isomorphism.
(ii) Given a finite cat^1-group $s, t\colon C \to Ob(C)$ we can search through the subgroups of C to find a smallest cat^1-group $K \subset C$ such that the inclusion $K \hookrightarrow C$ is a quasi-isomorphism.
(iii) We can repeatedly apply (i) followed by (ii).

The assertion that $C \to C/N$ is a quasi-isomorphism in Procedure 3(i) is readily checked.

For the dihedral group G of order 32 the crossed module $\partial\colon G \to Aut(G)$ is equivalent to a cat^1-group C^∂ whose underlying group has order 4096. By the above method we can construct a subgroup $H^\partial \le C^\partial$ of order 64 that gives rise to a quasi-isomorphism $H^\partial \to C^\partial$. The orders of the groups in the nerves of C^∂ and H^∂ are related by

$$|N(C^\partial)_k| = 8^{k+1}|N(H^\partial)_k|.$$

5. An idea for improving the homology computations

The chain complex constructed for a simplicial group using Theorem 1 and Procedure 3 is typically unnecessarily large. Given any chain complex (A_*, d) of finitely generated free abelian groups there are a number of ways in which one might attempt to produce a chain homotopy equivalence $A_* \simeq B_*$ where (B_*, d') is a chain complex of free abelian groups of lower ranks. We shall describe one such procedure which grew out of conversations with Pawel Dlotko, T. Kaczynski and Marian Mrozek and is based on ideas in their paper (9). It is extremely simple to implement yet surprisingly effective in many cases.

Let us denote by e_i^n the free generators of A_n. Let us define a pair (e_i^n, e_j^{n-1}) to be *redundant* if $d(e_i^n) = \pm e_j^{n-1}$. A redundant pair generates a sub chain complex $\langle e_i^n, e_j^{n-1}\rangle$ of A_* with trivial homology. The long exact homology sequence arising from a short exact sequence of chain complexes implies that the quotient chain map $\Pi_*\colon A_* \to A_*/\langle e_i^n, e_j^{n-1}\rangle$ induces isomorphisms on homology. Moreover, the quotient $A_*/\langle e_i^n, e_j^{n-1}\rangle$ is a chain complex of free abelian groups and hence Π_* must be a chain homotopy equivalence.

Procedure 4. Given a finite dimensional chain complex A_* we set $A_*^{(0)} = A_*$ and, while $A_*^{(k)}$ contains a redundant pair (e_i^n, e_j^{n-1}), we set $A_*^{(k+1)} = A_*^{(k)}/\langle e_i^n, e_j^{n-1}\rangle$. When $A_*^{(k)}$ contains no redundant pair we set $B_* = A_*^{(k)}$. The composite of chain homotopy equivalences

$$A_*^{(0)} \to A_*^{(1)} \to \cdots \to A_*^{(k)}$$

provides a chain homotopy equivalence $A_* \to B_*$.

An implementation of Procedure 4 is available in HAP (13). To illustrate its performance we consider the crossed module $\partial\colon \mathbb{Z}_2 \to 0$ representing an Eilenberg-Mac Lane space $K(\mathbb{Z}_2, 2)$. Let $K_* = C_*(N(C^\partial))$ denote the chain complex for the nerve of the cat^1-group C^∂ constructed using Theorem 1. When Procedure 4 is applied to this chain complex K_* it yields a chain homotopy equivalence $K_* \simeq L_*$ where the ranks of K_i and

L_i are listed in the following table for low degrees.

i	0	1	2	3	4	5	6	7	8	9	10
Rank(K_i)	1	1	2	4	8	16	32	64	128	256	512
Rank(L_i)	1	0	1	1	2	3	5	8	13	21	34

Experimental evidence seems to suggest that Procedure 4 yields a free abelian chain complex for $K(\mathbb{Z}_m, 2)$, $m \geq 2$, whose terms have ranks equal to the Fibonacci numbers. We remark that Clemens Berger (2) has proved the existence of a CW-complex of type $K(\mathbb{Z}_2, 2)$ whose terms have ranks equal to the Fibonacci numbers.

We now come to the idea for improving our procedure for computing homology of n-types. Suppose that

$$1 \to N_* \to G_* \to Q_* \to 1 \tag{6}$$

is a short exact sequence (i.e. *fibration sequence*) of simplicial groups. Using the perturbation technique explained in Wall's paper (29) the HAP package (13) can produce effective bar complexes $(\overline{R}_*^{G_p}, \overline{B}_*^{G_p}, \iota, \psi, H)$ in which each chain complex $\overline{R}_*^{G_p}$ has the form of a *twisted tensor product* $\overline{R}_*^{G_p} = \overline{R}_*^{N_p} \tilde{\otimes} \overline{R}_*^{Q_p}$. The free abelian group underlying $\overline{R}_*^{N_p} \tilde{\otimes} \overline{R}_*^{Q_p}$ is the same as that underlying the usual tensor product of chain complexes $\overline{R}_*^{N_p} \otimes \overline{R}_*^{Q_p}$; the differential on the twisted tensor product is a perturbation of the usual differential. The chain complex KG_* constructed for the simplicial group G_* using Theorem 1 thus has the form of a twisted tensor product

$$KG_* = KN_* \tilde{\otimes} KQ_*$$

(whose underlying free abelian group coincides with that of $KG_* = KN_* \otimes KQ_*$).

Suppose now that for the chain complexes KN_* and KQ_* produced using Theorem 1 we have used Procedure 4 to obtain homotopy equivalence data

$$(LN_*, d) \overset{\psi}{\underset{\iota}{\leftrightarrows}} (KN_*, d),\ H, \tag{7}$$

$$(LQ_*, d) \overset{\psi}{\underset{\iota}{\leftrightarrows}} (KQ_*, d),\ H. \tag{8}$$

We can combine (7) and (8) to produce a homotopy equivalence data

$$(LN_* \otimes LQ_*, d) \overset{\psi}{\underset{\iota}{\leftrightarrows}} (KN_* \otimes KQ_*, d),\ H. \tag{9}$$

Lemma 2 can be applied to a perturbation on the right hand side of (9) to obtain a homotopy equivalence data

$$(LN_* \tilde{\otimes} LQ_*, d) \overset{\psi}{\underset{\iota}{\leftrightarrows}} (KN_* \tilde{\otimes} KQ_*, d),\ H. \tag{10}$$

The left hand side of (10) is a small chain complex for computing the homology of G_*. Note that by using a lazy evaluation programming style, an implementation of the chain complex $LN_* \tilde{\otimes} LQ_*$ does not require all differentials in KG_* to be computed.

We have only implemented this improvement for the easy case where $G_* = N_* \times Q_*$ in the fibration sequence (6). As an illustration let us take $G = C_3 \rtimes C_8$ to be the first group of order 24 in the GAP small groups library. The crossed module $\partial\colon G \to Aut(G)$ corresponds to a cat^1-group C^∂ of order 576. There is a quasi-isomorphism $C^\partial \simeq D^\partial$ with D^∂ a cat^1-group of order 16. Now $\pi_1(\partial) = C_2 \times C_2$ and $\pi_2(\partial) = C_4$. By construction $\pi_1(\partial)$ acts trivially on $\pi_2(\partial)$. It follows that C^∂ represents the homotopy type $K(C_4, 2) \times K(C_2 \times C_2, 1)$. Using Theorem 1 and Procedure 4 we can compute a chain complex LN_* for $K(C_4, 2)$. We can compute a minimal chain complex LQ_* for $K(C_2 \times C_2, 1)$. The tensor product $LN_* \otimes LQ_*$ yields the homology of the crossed module ∂. Ranks for these chain complexes in low degrees are given in the following table.

i	0	1	2	3	4	5	6
Rank(LN_i)	1	0	1	3	8	19	46
Rank(LQ_i)	1	2	3	4	5	6	7
Rank$((LN_* \otimes LQ_*)_i)$	1	2	4	9	22	54	132

We have $H_5(C^\partial, \mathbb{Z}) = H_5(LN_* \otimes LQ_*) = (\mathbb{Z}_2)^{11}$.

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