The Mod-2 Cohomology Ring of the Third Conway Group is Cohen–Macaulay

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By explicit machine computation we obtain the mod-2 cohomology ring of the third Conway group Co_3. It is Cohen-Macaulay, has dimension 4, and is detected on the maximal elementary abelian 2-subgroups.

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1 Introduction

There has been considerable work on the mod-2 cohomology rings of the finite simple groups. Every finite simple group of 2-rank at most three has Cohen–Macaulay mod-2 cohomology [2]. There are eight sporadic finite simple groups of 2-rank four. For six of these, the mod-2 cohomology ring has already been determined, at least as a module over a polynomial subalgebra [3, VIII.5]. In most cases the cohomology is not Cohen–Macaulay. For instance, the Mathieu groups M_{22} and M_{23} each have maximal elementary abelian 2-subgroups of ranks 3 and 4, meaning that the cohomology cannot be Cohen–Macaulay: see [3, p. 269].

The two outstanding cases have the largest Sylow 2-subgroups. The Higman–Sims group HS has size 2^9 Sylow subgroup, and the cohomology of this 2-group is known [1]. The third Conway group Co_3 has size 2^{10} Sylow subgroup.

In this paper we consider Co_3. It stands out for two reasons, one being that it has the largest Sylow 2-subgroup. The second reason requires a little explanation. The Mathieu group M_{12} has 2-rank three. Milgram observed that 2-locally it looks as if M_{12} admits a faithful representation in the Lie group G_2, but that is impossible. Benson and Wilkerson made this more precise [10] by constructing a map of classifying spaces with good properties in mod-2 cohomology.

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Benson took a similar approach to Co_3. After 2-completion, its classifying space admits a map to that of DI(4). This is a monomorphism in mod-2 cohomology, and $H^*(Co_3, \mathbb{F}_2)$ is finitely generated as a module over its image [4].

The Dwyer–Wilkerson exotic finite loop space DI(4) has the rank four Dickson invariants as its mod-2 cohomology [15]. So Benson's result says that the Dickson invariants form a homogeneous system of parameters for $H^*(Co_3, \mathbb{F}_2)$ in degrees $8, 12, 14, 15$. Benson asks if these parameters form a regular sequence [5]. That is, he suggests that $H^*(Co_3, \mathbb{F}_2)$ might be Cohen–Macaulay. Certainly the Dickson invariants constitute a filter-regular system of parameters [8, Thm 1.2].

By a mixture of machine computation and theoretical argument we obtain the following theorem, answering Benson's question in the affirmative:

Theorem 1.1 *The mod-2 cohomology ring $H^*(Co_3, \mathbb{F}_2)$ of the third Conway group Co_3 has the following properties:*

1. *As a commutative $\mathbb{F}_2$-algebra, it has 16 generators and 71 relations. A full presentation is given in Appendix A. The smallest generator degree is 3, and the greatest is 15. The greatest degree of a relation is 33.*
2. *It is Cohen–Macaulay, having Krull dimension 4 and depth 4.*
3. *It has zero nilradical, and is detected on the maximal elementary abelian 2-subgroups. These all have rank 4, and form four conjugacy classes.*
4. *Its Poincaré series is of the form*
$$P(t) = \frac{f(t)}{(1-t^8)(1-t^{12})(1-t^{14})(1-t^{15})},$$
where $f(t) \in \mathbb{Z}[t]$ is the monic polynomial of degree 45 with the coefficients 1, 1, 1, 1, 2, 3, 3, 4, 4, 6, 7, 8, 9, 10, 10, 11, 13, 12, 14, 15, 13, 13, 15, 14, 12, 13, 11, 10, 10, 9, 8, 7, 6, 4, 4, 3, 3, 2, 1, 1, 1, 1.

Remark 1.2 As we can hardly expect each reader to write their own program to check our computational results, it is highly desirable to have some consistency checks for the final result. Benson–Carlson duality [9, Thm 1.1] provides one. It states that if a cohomology ring is Cohen–Macaulay, then it is Gorenstein in the graded sense with a-invariant zero.

We find that $H^*(Co_3, \mathbb{F}_2)$ is Cohen–Macaulay, and recover Benson's result that the Dickson invariants form a system of parameters. Hence Benson–Carlson duality requires the numerator $f(t)$ in the above Poincaré series to be symmetric of degree $45 = 7 + 11 + 13 + 14$, in the sense that the coefficients remain the same when read from back to front. Observe that this is indeed the case.

We computed the cohomology of the Sylow subgroup using our package [21]. Then we computed the stable elements degree by degree, following Holt [19]. We used our variant [18, Thm 3.3] of Benson's test [8] to tell when to stop.

Remark 1.3 We actually constructed Benson's Dickson invariants in $H^*(Co_3, \mathbb{F}_2)$, in order to obtain an explicit filter regular system of parameters.

Structure of the paper

We recall the stable elements method in Section 2, discussing how to reduce the number of stability checks. In Section 3 we consider how to implement stability checks and Benson's test for non-p-groups. We highlight the relevant group theory of Co_3 in Section 4, proving Theorem 1.1.

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2 Stable elements

Let p be a prime, G a finite group, and $H \le G$ a subgroup whose index is coprime to p. Following Holt [19, p. 352] we compute $H^*(G, \mathbb{F}_p)$ as the ring of stable elements [12, XII § 10] in $H^*(H, \mathbb{F}_p)$. Recall that $x \in H^*(H, \mathbb{F}_p)$ is stable if

$$\forall g \in G \quad \mathrm{Res}^H_{H^g \cap H}(x) = g^* \,\mathrm{Res}^H_{H \cap {}^g H}(x)\,, \quad \text{where } g^* = c_g^* \text{ for } c_g(h) = ghg^{-1}.$$

Note that H need not be a Sylow subgroup [6, Prop. 3.8.2]. The stability condition associated to g only depends on the double coset $HgH \in H \backslash G / H$.

Using intermediate subgroups

Let S be a Sylow p-subgroup of the finite group G. Holt observed that the total number of stability conditions is reduced dramatically if one works up a tower of subgroups

$$S = G_0 \le G_1 \le G_2 \le \cdots \le G_n = G\,,$$

where each $|G_i \colon G_{i-1}|$ is as small as possible. One determines $H^*(G_i, \mathbb{F}_p)$ as the ring of stable elements in $H^*(G_{i-1}, \mathbb{F}_p)$. Often we take $G_1 = N_G(Z(S))$.

Discarding double cosets

For some double cosets the associated stability condition is satisfied by every $x \in H^*(H, \mathbb{F}_p)$. Such double cosets can be discarded.

For example, the trivial double coset $H1H$ can always be discarded. And HgH can be discarded if $H^g \cap H$ has order coprime to p. Proposition 18 of [17] generalizes to a group-theoretic criterion for the redundancy of some double cosets.

Lemma 2.1 *Let $H \leq G$ be a subgroup with p' index. Let $g \in G$, and let T be a Sylow p-subgroup of $H^g \cap H$. Suppose that transfer from $H^*(T, \mathbb{F}_p)$ to $H^*(G, \mathbb{F}_p)$ is the zero map. Then the stability condition associated to HgH is redundant.*

In particular if there is a p-group $W \neq 1$ such that $T \times W \leq G$, then the stability condition associated to HgH is redundant.

See Remark 4.2 for an application of this result.

Proof We do not claim that the stability condition is always satisfied. The proof of the stable elements method in [6, Prop. 3.8.2] uses a weaker condition: that stability holds after transfer from $H^g \cap H$ to G. So if the transfer map is zero, then the double coset is redundant. But transfer from $H^g \cap H$ factors through transfer from T to G, since transfer from T to $H^g \cap H$ is a split surjection.

Last part: Transfer from T to G factors through transfer from T to $T \times W$, which is zero: for restriction from $T \times W$ to T is a split surjection, and restriction followed by transfer is multiplication by $|W|$. □

To perform the stability test for HgH we first construct the induced homomorphisms $\mathrm{Res}^H_{H^g \cap H}$ and $g^* \mathrm{Res}^H_{H \cap {}^g H}$, determining the images of the ring generators. If each generator has the same image both times then we discard the double coset. Similarly, we discard it if the pair of maps has been seen already. This too saves effort, for the most time-intensive step is the next one: working out the matrices of the two linear maps from $H^n(H, \mathbb{F}_p)$ to $H^n(H^g \cap H, \mathbb{F}_p)$ degree by degree.

3 Computational aspects

Representing cohomology rings

We consider how to represent the cohomology ring of a finite group on the computer. Reusing the results of previous computations saves time, but it does involve coherence issues.

Let G be a finite group and $S \leq G$ a Sylow p-subgroup. We assume that we already know the cohomology of a group $\bar{S}$ isomorphic to S. In order to make use of this computation we choose an isomorphism $f\colon \bar{S} \to S$. We can then store $H^*(G, \mathbb{F}_p)$ by recording the map f together with the image ring $R_{G,f}$ given by

$$R_{G,f} = f^* \left(\operatorname{Res}^G_S H^*(G, \mathbb{F}_p) \right) \subseteq H^*(\bar{S}, \mathbb{F}_p)\,.$$

Now suppose that $\phi\colon G_1 \to G_2$ is a group homomorphism, and that we calculated $H^*(G_i, \mathbb{F}_p)$ for $i = 1, 2$ using the Sylow p-subgroup S_i and the isomorphism $f_i\colon \bar{S}_i \to S_i$. We represent ϕ^* as the composition

$$R_{G_2,f_2} \xrightarrow{\cong} H^*(G_2, \mathbb{F}_p) \xrightarrow{\phi^*} H^*(G_1, \mathbb{F}_p) \xrightarrow{\cong} R_{G_1,f_1}\,.$$

As $\phi(S_1)$ is a p-subgroup of G_2, we may pick $g \in G_2$ such that $\phi'(S_1) \leq S_2$, where $\phi' = c_g \circ \phi$. Then $\phi'^* = \phi^*$, since c_g is an inner automorphism of G_2. Let $\bar{\phi}\colon \bar{S}_1 \to \bar{S}_2$ be the homomorphism $\bar{\phi} = f_2^{-1} \circ \phi' \circ f_1$. Then $\bar{\phi}^*$ maps $R_{G_2,f_2} \subseteq H^*(\bar{S}_2, \mathbb{F}_p)$ to $R_{G_1,f_1} \subseteq H^*(\bar{S}_1, \mathbb{F}_p)$ in the desired way.

Stability and the representation

Let $S \leq H \leq G$, where S is Sylow in G and $H^*(H, \mathbb{F}_p)$ is known: so we know $R_{H,f}$ for an isomorphism $f\colon \bar{S} \to S$. The stability test for HgH asks for the equalizer of $\phi_1^*, \phi_2^*\colon H^*(H, \mathbb{F}_p) \to H^*(H^g \cap H, \mathbb{F}_p)$, where $\phi_1, \phi_2\colon H^g \cap H \to H$ are $\phi_1(h) = h$ and $\phi_2(h) = ghg^{-1}$.

Typically the cohomology of $H^g \cap H$ will not yet be known, but the cohomology of its Sylow subgroup T will be. We have two options:

- We compute $H^*(H^g \cap H, \mathbb{F}_p)$ and construct ϕ_1^*, ϕ_2^* as above.
- We take the equalizer of $\psi_1^*, \psi_2^*\colon H^*(H, \mathbb{F}_p) \to H^*(T, \mathbb{F}_p)$ instead, where $\psi_i = \phi_i|_T$. This works since $\operatorname{Res}^{H^g \cap H}_T$ is injective.

To our surprise, the first method proved to be more efficient. One possible explanation is that $H^n(H^g \cap H, \mathbb{F}_p)$ often has considerably smaller dimension than $H^n(T, \mathbb{F}_p)$. This reduces the size of the matrices representing the two maps: and matrix size seems to have the greatest influence on running time.

Remark Holt [19] chooses good double coset representatives at the outset. We are effectively taking the first ones we find and correcting them later on.

Computing stable elements degree by degree

We have translated each stability check into taking the equalizer of two known ring homomorphisms. We now have to determine the equalizers and then take their intersection. One approach would be efficient algorithms for ideals, though we might have to implement these ourselves. Another would be to compute parameters for $H^*(G, \mathbb{F}_p)$ using e.g. Chern classes, and then to use algorithms for noetherian modules.

We take a different approach and work degree by degree. Then performing a stability check just means taking the nullspace of a matrix. This is easier to implement, but linear algebra on its own cannot tell when to stop.

Following Benson, we write $\tau_d H^*(G, \mathbb{F}_p)$ for the $\mathbb{F}_p$-algebra generated by the indecomposable elements of $H^*(G, \mathbb{F}_p)$ in degree $\leq d$, subject to the relations which hold in $H^*(G, \mathbb{F}_p)$ in degrees $\leq d$. Assume that we already have $\tau_{d-1} H^*(G; \mathbb{F}_p)$, and have recorded the image in $H^*(H, \mathbb{F}_p)$ of each generator. So we can construct the image in $H^d(H, \mathbb{F}_p)$ of each degree d standard monomial of $\tau_{d-1} H^*(G, \mathbb{F}_p)$. A degree d relation in $H^*(G, \mathbb{F}_p)$ corresponds to a linear dependence between these images; and if the images do not span the subspace of stable elements in $H^d(H, \mathbb{F}_p)$, then we get new generators. This determines $\tau_d H^*(G, \mathbb{F}_p)$.

Remark The third author has implemented the stable elements method in his HAP system. With Dutour Sikirić he used it to compute the integral homology of the Mathieu group M_{24} out to degree four [14].

Constructing filter regular parameters

We use Benson's test for completion [8, Thm 10.1] to tell when d is large enough to ensure that $\tau_d H^*(G, \mathbb{F}_p) = H^*(G, \mathbb{F}_p)$. The key step is to construct homogeneous elements $h_1, \ldots, h_r \in \tau_d H^*(G, \mathbb{F}_p)$ which are a filter-regular system of parameters for both $\tau_d H^*(G, \mathbb{F}_p)$ and $H^*(G, \mathbb{F}_p)$. Here, $r = p\text{-rk}(G)$. We need one technical result.

Lemma 3.1 *Suppose that $c_1, \ldots, c_n \in H^*(G, \mathbb{F}_p)$ is a filter-regular sequence in $H^*(S, \mathbb{F}_p)$. Then it is filter-regular in $H^*(G, \mathbb{F}_p)$ too.*

Proof $H^*(G, \mathbb{F}_p)$ is a direct summand of the $H^*(G, \mathbb{F}_p)$-module $H^*(S, \mathbb{F}_p)$, by virtue of the transfer map. The result follows. □

Assume that d is large enough, so that $H^*(G, \mathbb{F}_p)$ is finite over $\tau_d H^*(G, \mathbb{F}_p)$. By [8, Coroll. 9.8] there are filter-regular parameters $d_1, \ldots, d_r$ which restrict to each maximal elementary abelian p-subgroup as (powers of) the Dickson invariants.

Parameters in low degrees allow us to terminate the computation earlier. The Dickson invariants are in rather high degree. Sections 2 and 3 of [18] present several ways of lowering the degrees. One of these methods can however fail for non-p-groups: the weak rank-restriction condition [18, Lemma 2.3]. So we proceed as follows. Set $z = p\text{-rk}(Z(S))$.

(1) Construct the Dickson invariants $d_1, ..., d_r$ if this is not too difficult.

(2) Using $d_1, \ldots, d_z$ or otherwise, find $c_1, \ldots, c_z \in \tau_d H^*(G, \mathbb{F}_p)$ which restrict to parameters for $H^*(Z(S), \mathbb{F}_p)$. For non-p-groups there is no guarantee that these c_i may be chosen from among the ring generators.

(3) Using [18, Lemma 2.3], find filter-regular parameters $c_1, \ldots, c_r$ for $H^*(S, \mathbb{F}_p)$, extending $c_1, \ldots, c_z$. If $c_{z+1}, \ldots, c_r$ are stable then $c_1, \ldots, c_r$ is filter-regular for $H^*(G, \mathbb{F}_p)$ by Lemma 3.1.

(4) If only c_r fails stability, then replace it by any stable class that finishes off the parameter system.

(5) Use the factorization and nilpotent alteration methods (Lemmas 2.5 and 2.7 of [18]) to reduce the degrees of $d_1, \ldots, d_r$ and/or $c_1, \ldots, c_r$.

With luck we thus construct a filter-regular system of parameters and can compute its filter-degree type. Benson's test then gives us a degree bound involving the sum of the parameter degrees. If this is too large then we use the existence result [18, Prop. 3.2] for low-degree parameters over an extension field in order to apply our variant of Benson's test [18, Thm 3.3].

4 The third Conway group

The Sylow 2-subgroup

The third Conway group Co_3 is simple and admits a degree 276 faithful permutation representation [13]. The Sylow 2-subgroups have order 2^{10}. The Online ATLAS [23] contains explicit permutations for the degree 276 representation. GAP [16] easily constructs the Sylow 2-subgroup S.

Despite its size, computing $H^*(S, \mathbb{F}_2)$ is a surprisingly routine application of our program [21]. The result may be viewed online [20]. Duflot's lower bound for the depth [11, Thm 12.3.3] is one, and the Krull dimension is four. In fact the depth is three. This led Dave Benson to reiterate to us his conjecture that $H^*(Co_3, \mathbb{F}_2)$ could be Cohen–Macaulay.

The maximal elementary abelian subgroups

There are two conjugacy classes of involutions in Co_3: classes 2A and 2B with centralizer sizes 2,903,040 and 190,080 respectively. Using GAP one sees that Co_3 has four conjugacy classes of maximal elementary abelian 2-subgroups. Each has rank 4, and they are distinguished by the number of 2A elements they contain. In ATLAS notation:

$$V_1 = 2A_1B_{14} \qquad V_2 = 2A_3B_{12} \qquad V_3 = 2A_7B_8 \qquad V_4 = 2A^4 \,.$$

For each $1 \leq r \leq 4$ there is a subgroup $2A^r \leq V_r$ containing all the 2A elements.

A tower of subgroups

The Sylow 2-subgroup has 484,680 double cosets in Co_3. It is therefore essential that we find a convenient tower of subgroups.

The order 4 elements in Co_3 form two conjugacy classes [13]. Type 4A elements have size 23,040 centralizer, and type 4B elements have size 1,536 centralizer.

Lemma 4.1 *Let S be a Sylow 2-subgroup of $G = Co_3$. Then*

(1) *The centre $Z(S)$ and the second centre $Z_2(S)$ have isomorphism types $Z(S) \cong C_2$ and $Z_2(S) \cong C_4 \times C_2$.*

(2) *$Z_2(S)$ has Frattini subgroup $Z(S)$. So does each copy of C_4 in $Z_2(S)$.*

(3) *Precisely one subgroup $U \leq Z_2(S)$ is generated by a type $4A$ element.*

(4) *$N_G(Z_2(S)) \leq N_G(U) \leq N_G(Z(S))$.*

Proof The first two are easily checked in GAP [16] using the permutation representation. For the third statement one inspects the four order 4 elements in $Z_2(S) \cong C_4 \times C_2$, finding two of type 4A, and two of type 4B. The centralizer sizes differ, so the two type 4A elements lie in the same cyclic subgroup.

The last part now follows, for $Z(S)$ is a characteristic subgroup of U, and no other subgroup of $Z_2(S)$ is conjugate to U in $G = Co_3$. □

Consider the tower of subgroups $S = G_0 \leq G_1 \leq G_2 \leq G_3 \leq G_4 = Co_3$ given by

$$G_1 = N_G(Z_2(S)) \qquad G_2 = N_G(U) \qquad G_3 = N_G(Z(S))\,.$$

G_3 is a maximal subgroup of Co_3 [13]. The sizes of the layers are as follows:

i	$\lvert G_i : G_{i-1}\rvert$	$\lvert G_{i-1}\backslash G_i/G_{i-1}\rvert$
1	3	2
2	15	3
3	63	3
4	$170,775$	7

As the trivial double coset can be discarded, working up the tower involves a total of $1+2+2+6=11$ stability conditions.

Remark 4.2 We can discard 4 more double cosets when computing $H^*(Co_3,\mathbb{F}_2)$ from $H^*(G_3,\mathbb{F}_2)$. Every maximal elementary abelian has rank 4, so if the Sylow subgroup of $G_3^g \cap G_3$ is elementary abelian of rank ≤ 3, then Lemma 2.1 applies with $W = C_2$. There are three double cosets where the Sylow subgroup is elementary abelian of order 4, and one where it is cyclic of order 2.

Proof of Theorem 1.1 We computed the mod-2 cohomology ring of the Sylow subgroup using our package [21]. We then used the stable elements method and the computational methods of Section 3 to work up the tower of subgroups.

The depth is a by-product of a computation based on Benson's test. The depth of $H^*(G_i,\mathbb{F}_2)$ is weakly increasing in i: see [7, Thm 2.1], and note that the proof only requires the index to be coprime to p. We remarked that $H^*(G_0,\mathbb{F}_2)$ already has depth 3. It turns out that $H^*(G_1,\mathbb{F}_2)$ has depth 4. So $H^*(G_i,\mathbb{F}_2)$ is Cohen–Macaulay for all $i \geq 1$. Thus we established (1), (2) and (4).

For (3): The depth is 4, and so by a result of Carlson [11, Thm 12.5.2] the centralizers of the rank four elementary abelians detect $H^*(Co_3,\mathbb{F}_2)$. We saw above that there are four conjugacy classes of rank four elementary abelians. Using GAP one sees that each is self-centralizing. And the nilradical vanishes, as elementary abelian 2-groups have polynomial cohomology. □

Report on filter-regular parameters

The 2-rank of Co_3 is four, so the Dickson elements are in degrees 8, 12, 14 and 15 for any subgroup in the tower. As $H^*(Co_3,\mathbb{F}_2)$ contains these Dickson invariants [4], so does each $H^*(G_i,\mathbb{F}_2)$: no higher powers are necessary.

G_0 is the Sylow subgroup, with rank one centre. Applying the weak rank-restriction condition [18, Lemma 2.3] we constructed filter-regular parameters c_1, c_2, c_3, c_4 in degrees 8, 4, 6 and 7. Using [18, Prop. 3.2] we demonstrated the existence of filter-regular parameters in degrees 8, 4, 2 and 2. This allowed us to terminate the calculation in degree 14, where the last relation is found.

Our c_1, c_2, c_3 are stable for G_3, and so c_1, c_2, c_3 is a filter-regular sequence in $H^*(G_i, \mathbb{F}_2)$ for $i = 1, 2, 3$. For $i = 1, 2$ we found a fourth parameter in degree 1, so the calculations for $H^*(G_1, \mathbb{F}_2)$ and $H^*(G_2, \mathbb{F}_2)$ terminate when the last relation is found in degree 16. For G_3 we found a fourth parameter in degree 7, detecting completion in degree 21. The presentation is complete after degree 18.

For $G_4 = Co_3$ we had to construct Benson's Dickson invariants, detecting completion in degree 45. The presentation is complete after degree 33.

A A Minimal Ring Presentation

Ring generators are denoted by a letter with two indices. $H^*(Co_3, \mathbb{F}_2)$ has no nilradical. The letter 'b' denotes a generator with nilpotent restriction to the centre $Z(S)$ of the Sylow subgroup. The letter 'c' denotes a *Duflot element*, whose restriction to $Z(S)$ is non-nilpotent. The first index gives the degree of the generator, the second is to distinguish generators of the same degree. This presentation is also available online [20].

A minimal generating set for $H^*(Co_3; \mathbb{F}_2)$ is given by

$$b_{4,0}, b_{6,1}, b_{8,1}, c_{8,3}, b_{12,1}, b_{12,7}, b_{14,1}, b_{3,0}, b_{5,0}, b_{7,0}, b_{7,1}, b_{9,0}, b_{11,5}, b_{13,1}, b_{13,7}, b_{15,13}.$$

The following polynomials form a minimal generating set of the relation ideal:

(1) $b_{5,0}^2 + b_{3,0}b_{7,0} + b_{4,0}b_{6,1}$

(2) $b_{3,0}^2 b_{5,0} + b_{8,1}b_{3,0} + b_{4,0}b_{7,1}$

(3) $b_{3,0}b_{9,0} + b_{4,0}b_{8,1} + b_{4,0}^3$

(4) $b_{5,0}b_{7,0} + b_{3,0}^4 + b_{6,1}b_{3,0}^2 + b_{6,1}^2 + b_{4,0}^3$

(5) $b_{5,0}b_{7,1} + b_{4,0}b_{8,1} + b_{4,0}^3$

(6) $b_{6,1}b_{7,1} + b_{4,0}b_{9,0} + b_{4,0}b_{6,1}b_{3,0} + b_{4,0}^2 b_{5,0}$

(7) $b_{3,0}^2 b_{7,0} + b_{8,1}b_{5,0} + b_{4,0}b_{9,0} + b_{4,0}b_{6,1}b_{3,0}$

(8) $b_{3,0}^2 b_{7,1} + b_{4,0}b_{9,0} + b_{4,0}b_{3,0}^3 + b_{4,0}^2 b_{5,0}$

(9) $b_{6,1}b_{3,0}b_{5,0} + b_{6,1}b_{8,1} + b_{4,0}b_{3,0}b_{7,1} + b_{4,0}^2b_{3,0}^2 + b_{4,0}^2b_{6,1}$

(10) $b_{5,0}b_{9,0} + b_{4,0}b_{3,0}b_{7,1} + b_{4,0}b_{3,0}b_{7,0} + b_{4,0}^2b_{3,0}^2 + b_{4,0}^2b_{6,1}$

(11) $b_{7,0}b_{7,1} + b_{4,0}b_{3,0}b_{7,0}$

(12) $b_{6,1}b_{9,0} + b_{4,0}b_{6,1}b_{5,0} + b_{4,0}^2b_{7,1} + b_{4,0}^3b_{3,0}$

(13) $b_{12,7}b_{3,0} + b_{4,0}b_{11,5} + b_{4,0}c_{8,3}b_{3,0}$

(14) $b_{3,0}^5 + b_{8,1}b_{7,0} + b_{6,1}b_{3,0}^3 + b_{6,1}^2b_{3,0} + b_{4,0}^2b_{7,0} + b_{4,0}^3b_{3,0}$

(15) $b_{3,0}b_{13,1} + b_{8,1}b_{3,0}b_{5,0} + b_{8,1}^2 + b_{4,0}^2b_{8,1}$

(16) $b_{3,0}b_{13,7} + b_{4,0}b_{12,7} + c_{8,3}b_{3,0}b_{5,0}$

(17) $b_{5,0}b_{11,5} + b_{4,0}b_{12,7} + c_{8,3}b_{3,0}b_{5,0}$

(18) $b_{7,0}b_{9,0} + b_{4,0}b_{3,0}^4 + b_{4,0}b_{6,1}b_{3,0}^2 + b_{4,0}b_{6,1}^2 + b_{4,0}^4$

(19) $b_{7,1}b_{9,0} + b_{8,1}b_{3,0}b_{5,0} + b_{8,1}^2 + b_{4,0}^2b_{3,0}b_{5,0} + b_{4,0}^2b_{8,1}$

(20) $b_{6,1}b_{11,5} + b_{4,0}b_{13,7} + b_{6,1}c_{8,3}b_{3,0} + b_{4,0}c_{8,3}b_{5,0}$

(21) $b_{8,1}b_{9,0} + b_{4,0}b_{13,1} + b_{4,0}b_{8,1}b_{5,0}$

(22) $b_{12,7}b_{5,0} + b_{4,0}b_{13,7} + b_{4,0}c_{8,3}b_{5,0}$

(23) $b_{3,0}^2b_{11,5} + b_{4,0}b_{13,7} + c_{8,3}b_{3,0}^3 + b_{4,0}c_{8,3}b_{5,0}$

(24) $b_{3,0}b_{7,1}^2 + b_{4,0}b_{13,1} + b_{4,0}^2b_{3,0}^3$

(25) $b_{6,1}b_{12,7} + b_{4,0}b_{3,0}b_{11,5} + b_{4,0}c_{8,3}b_{3,0}^2$

(26) $b_{3,0}b_{15,13} + b_{6,1}b_{12,1} + b_{4,0}b_{7,1}^2 + b_{4,0}b_{7,0}^2 + b_{4,0}b_{14,1} + b_{4,0}^3b_{3,0}^2 + c_{8,3}b_{3,0}b_{7,0}$

(27) $b_{5,0}b_{13,1} + b_{4,0}b_{7,1}^2 + b_{4,0}^3b_{3,0}^2$

(28) $b_{5,0}b_{13,7} + b_{4,0}b_{3,0}b_{11,5} + c_{8,3}b_{3,0}b_{7,0} + b_{4,0}c_{8,3}b_{3,0}^2 + b_{4,0}b_{6,1}c_{8,3}$

(29) $b_{7,0}b_{11,5} + c_{8,3}b_{3,0}b_{7,0}$

(30) $b_{9,0}^2 + b_{4,0}b_{7,1}^2 + b_{4,0}^2b_{3,0}b_{7,0} + b_{4,0}^3b_{3,0}^2 + b_{4,0}^3b_{6,1}$

(31) $b_{6,1}b_{13,1} + b_{4,0}b_{8,1}b_{7,1} + b_{4,0}^2b_{8,1}b_{3,0}$

(32) $b_{6,1}b_{13,7} + b_{4,0}^2b_{11,5} + b_{6,1}c_{8,3}b_{5,0} + b_{4,0}^2c_{8,3}b_{3,0}$

(33) $b_{12,1}b_{7,0}$

(34) $b_{12,7}b_{7,0}$

(35) $b_{12,7}b_{7,1} + b_{8,1}b_{11,5} + b_{4,0}^2b_{11,5} + b_{8,1}c_{8,3}b_{3,0} + b_{4,0}^2c_{8,3}b_{3,0}$

(36) $b_{14,1}b_{5,0} + b_{8,1}^2b_{3,0} + b_{6,1}b_{8,1}b_{5,0} + b_{6,1}^2b_{7,0} + b_{4,0}b_{15,13} + b_{4,0}b_{12,1}b_{3,0} +$
$b_{4,0}b_{6,1}b_{3,0}^3 + b_{4,0}b_{6,1}^2b_{3,0} + b_{4,0}^2b_{6,1}b_{5,0} + b_{4,0}^3b_{7,1} + b_{4,0}^3b_{7,0} + b_{4,0}c_{8,3}b_{7,0}$

(37) $b_{14,1}b_{3,0}^2 + b_{12,1}b_{3,0}b_{5,0} + b_{8,1}b_{3,0}^4 + b_{6,1}b_{7,0}^2 + b_{6,1}b_{14,1} + b_{6,1}b_{8,1}b_{3,0}^2 + b_{6,1}^2b_{8,1} +$
$b_{4,0}b_{6,1}b_{3,0}b_{7,0} + b_{4,0}^2b_{3,0}^4 + b_{4,0}^2b_{12,1} + b_{4,0}^2b_{6,1}b_{3,0}^2 + b_{4,0}^2b_{6,1}^2 + b_{4,0}^3b_{8,1} + b_{4,0}^5$

(38) $b_{5,0}b_{15,13} + b_{12,1}b_{3,0}b_{5,0} + b_{6,1}b_{7,0}^2 + b_{6,1}b_{14,1} + b_{4,0}b_{8,1}b_{3,0}b_{5,0} + b_{4,0}b_{8,1}^2 + b_{4,0}^3b_{8,1} + c_{8,3}b_{3,0}^4 + b_{6,1}c_{8,3}b_{3,0}^2 + b_{6,1}^2c_{8,3} + b_{4,0}^3c_{8,3}$

(39) $b_{7,0}b_{13,1}$

(40) $b_{7,0}b_{13,7} + c_{8,3}b_{3,0}^4 + b_{6,1}c_{8,3}b_{3,0}^2 + b_{6,1}^2c_{8,3} + b_{4,0}^3c_{8,3}$

(41) $b_{7,1}b_{13,7} + b_{8,1}b_{12,7} + b_{4,0}^2b_{12,7} + b_{4,0}b_{8,1}c_{8,3} + b_{4,0}^3c_{8,3}$

(42) $b_{9,0}b_{11,5} + b_{8,1}b_{12,7} + b_{4,0}^2b_{12,7} + b_{4,0}b_{8,1}c_{8,3} + b_{4,0}^3c_{8,3}$

(43) $b_{12,1}b_{3,0}^3 + b_{6,1}b_{15,13} + b_{6,1}b_{12,1}b_{3,0} + b_{4,0}b_{14,1}b_{3,0} + b_{4,0}b_{12,1}b_{5,0} + b_{4,0}b_{8,1}b_{3,0}^3 + b_{4,0}b_{6,1}b_{8,1}b_{3,0} + b_{4,0}b_{6,1}^2b_{5,0} + b_{4,0}^2b_{13,1} + b_{4,0}^2b_{6,1}b_{7,0} + b_{4,0}^3b_{3,0}^3 + b_{4,0}^3b_{6,1}b_{3,0} + b_{4,0}^4b_{5,0} + b_{6,1}c_{8,3}b_{7,0}$

(44) $b_{12,7}b_{9,0} + b_{8,1}b_{13,7} + b_{4,0}^2b_{13,7} + b_{8,1}c_{8,3}b_{5,0} + b_{4,0}^2c_{8,3}b_{5,0}$

(45) $b_{3,0}b_{7,1}b_{11,5} + b_{8,1}b_{13,7} + b_{4,0}^2b_{13,7} + b_{8,1}c_{8,3}b_{5,0} + b_{4,0}c_{8,3}b_{9,0} + b_{4,0}c_{8,3}b_{3,0}^3$

(46) $b_{7,0}^3 + b_{14,1}b_{7,0}$

(47) $b_{7,1}^3 + b_{8,1}b_{13,1} + b_{4,0}^2b_{13,1} + b_{4,0}^3b_{9,0} + b_{4,0}^3b_{3,0}^3 + b_{4,0}^4b_{5,0}$

(48) $b_{7,0}b_{15,13} + c_{8,3}b_{7,0}^2$

(49) $b_{7,1}b_{15,13} + b_{12,1}b_{3,0}b_{7,1} + b_{8,1}b_{7,1}^2 + b_{8,1}b_{14,1} + b_{8,1}^2b_{3,0}^2 + b_{6,1}b_{8,1}^2 + b_{6,1}^2b_{3,0}b_{7,0} + b_{4,0}b_{6,1}b_{3,0}^4 + b_{4,0}b_{6,1}^2b_{3,0}^2 + b_{4,0}^2b_{7,1}^2 + b_{4,0}^2b_{14,1} + b_{4,0}^2b_{8,1}b_{3,0}^2 + b_{4,0}^3b_{3,0}b_{7,0} + b_{4,0}^4b_{6,1} + b_{4,0}c_{8,3}b_{3,0}b_{7,0}$

(50) $b_{9,0}b_{13,1} + b_{8,1}b_{7,1}^2 + b_{4,0}^2b_{7,1}^2 + b_{4,0}^2b_{8,1}b_{3,0}^2 + b_{4,0}^4b_{3,0}^2$

(51) $b_{9,0}b_{13,7} + b_{4,0}b_{7,1}b_{11,5} + b_{4,0}c_{8,3}b_{3,0}b_{7,0} + b_{4,0}^2c_{8,3}b_{3,0}^2 + b_{4,0}^2b_{6,1}c_{8,3}$

(52) $b_{11,5}^2 + b_{12,1}b_{3,0}b_{7,1} + b_{8,1}b_{7,1}^2 + b_{8,1}b_{14,1} + b_{8,1}^2b_{3,0}^2 + b_{6,1}b_{8,1}^2 + b_{6,1}^2b_{3,0}b_{7,0} + b_{4,0}b_{6,1}b_{3,0}^4 + b_{4,0}b_{6,1}b_{12,1} + b_{4,0}b_{6,1}^2b_{3,0}^2 + b_{4,0}^2b_{7,0}^2 + b_{4,0}^2b_{8,1}b_{3,0}^2 + b_{4,0}^3b_{3,0}b_{7,0} + b_{4,0}^4b_{3,0}^2 + b_{4,0}^4b_{6,1} + c_{8,3}b_{7,1}^2 + b_{4,0}^2c_{8,3}b_{3,0}^2 + c_{8,3}^2b_{3,0}^2$

(53) $b_{12,7}b_{11,5} + b_{8,1}b_{15,13} + b_{6,1}b_{12,1}b_{5,0} + b_{4,0}^2b_{12,1}b_{3,0} + b_{8,1}c_{8,3}b_{7,1} + b_{8,1}c_{8,3}b_{7,0} + b_{4,0}c_{8,3}b_{11,5} + b_{4,0}b_{8,1}c_{8,3}b_{3,0} + b_{4,0}c_{8,3}^2b_{3,0}$

(54) $b_{14,1}b_{9,0} + b_{8,1}b_{15,13} + b_{8,1}^2b_{7,1} + b_{6,1}b_{12,1}b_{5,0} + b_{4,0}b_{12,1}b_{7,1} + b_{4,0}b_{6,1}b_{8,1}b_{5,0} + b_{4,0}b_{6,1}^2b_{7,0} + b_{4,0}^2b_{15,13} + b_{4,0}^2b_{12,1}b_{3,0} + b_{4,0}^2b_{6,1}b_{3,0}^3 + b_{4,0}^2b_{6,1}^2b_{3,0} + b_{4,0}^3b_{6,1}b_{5,0} + b_{4,0}^4b_{7,1} + b_{4,0}^4b_{7,0} + b_{8,1}c_{8,3}b_{7,0} + b_{4,0}^2c_{8,3}b_{7,0}$

(55) $b_{14,1}b_{3,0}b_{7,1} + b_{12,7}^2 + b_{4,0}b_{7,1}b_{13,1} + b_{4,0}b_{8,1}b_{3,0}^4 + b_{4,0}b_{8,1}b_{12,1} + b_{4,0}b_{6,1}b_{7,0}^2 + b_{4,0}b_{6,1}b_{14,1} + b_{4,0}b_{6,1}b_{8,1}b_{3,0}^2 + b_{4,0}b_{6,1}^2b_{8,1} + b_{4,0}^2b_{8,1}b_{3,0}b_{5,0} + b_{4,0}^2b_{8,1}^2 + b_{4,0}^2b_{6,1}b_{3,0}b_{7,0} + b_{4,0}^3b_{3,0}^4 + b_{4,0}^3b_{6,1}b_{3,0}^2 + b_{4,0}^3b_{6,1}^2 + b_{4,0}^6 + b_{8,1}c_{8,3}b_{3,0}b_{5,0} + b_{8,1}^2c_{8,3} + b_{4,0}^2b_{8,1}c_{8,3}$

(56) $b_{9,0}b_{15,13} + b_{12,7}^2 + b_{4,0}b_{12,1}b_{3,0}b_{5,0} + b_{4,0}b_{6,1}b_{7,0}^2 + b_{4,0}b_{6,1}b_{14,1} + b_{4,0}^2b_{8,1}b_{3,0}b_{5,0} + b_{4,0}^2b_{8,1}^2 + b_{4,0}^4b_{8,1} + b_{8,1}c_{8,3}b_{3,0}b_{5,0} + b_{8,1}^2c_{8,3} + b_{4,0}c_{8,3}b_{3,0}^4 + b_{4,0}b_{6,1}c_{8,3}b_{3,0}^2 + b_{4,0}b_{6,1}^2c_{8,3} + b_{4,0}^2b_{8,1}c_{8,3} + b_{4,0}^4c_{8,3}$

(57) $b_{11,5}b_{13,7} + b_{12,7}^2 + c_{8,3}^2 b_{3,0}b_{5,0}$

(58) $b_{12,7}b_{13,7} + b_{4,0}b_{14,1}b_{7,1} + b_{4,0}b_{12,1}b_{9,0} + b_{4,0}b_{8,1}b_{13,1} + b_{4,0}^2 b_{14,1}b_{3,0} + b_{4,0}^2 b_{12,1}b_{5,0} + b_{4,0}c_{8,3}b_{13,7} + b_{4,0}c_{8,3}b_{13,1} + b_{4,0}c_{8,3}^2 b_{5,0}$

(59) $b_{7,1}^2 b_{11,5} + b_{12,7}b_{13,1} + b_{4,0}^3 b_{13,7} + b_{4,0}c_{8,3}b_{13,1} + b_{4,0}^2 c_{8,3}b_{3,0}^3 + b_{4,0}^3 c_{8,3}b_{5,0}$

(60) $b_{11,5}b_{15,13} + b_{12,7}b_{14,1} + b_{12,1}b_{3,0}b_{11,5} + b_{8,1}b_{7,1}b_{11,5} + b_{4,0}^2 b_{7,1}b_{11,5} + b_{4,0}^3 b_{3,0}b_{11,5} + c_{8,3}b_{12,1}b_{3,0}^2 + b_{6,1}c_{8,3}b_{12,1} + b_{4,0}c_{8,3}b_{7,0}^2 + b_{4,0}c_{8,3}b_{14,1} + b_{4,0}b_{8,1}c_{8,3}b_{3,0}^2 + b_{4,0}^2 c_{8,3}b_{3,0}b_{7,1} + b_{4,0}^3 c_{8,3}b_{3,0}^2 + c_{8,3}^2 b_{3,0}b_{7,0}$

(61) $b_{13,1}^2 + b_{12,7}b_{14,1} + b_{12,1}b_{7,1}^2 + b_{12,1}b_{3,0}b_{11,5} + b_{8,1}b_{7,1}b_{11,5} + b_{4,0}^2 b_{7,1}b_{11,5} + b_{4,0}^2 b_{12,1}b_{3,0}^2 + b_{4,0}^3 b_{3,0}b_{11,5} + c_{8,3}b_{12,1}b_{3,0}^2 + b_{4,0}c_{8,3}b_{7,1}^2 + b_{4,0}b_{8,1}c_{8,3}b_{3,0}^2 + b_{4,0}^2 c_{8,3}b_{3,0}b_{7,1}$

(62) $b_{13,1}b_{13,7} + b_{8,1}b_{7,1}b_{11,5} + b_{4,0}^2 b_{7,1}b_{11,5} + b_{4,0}^3 b_{3,0}b_{11,5} + b_{4,0}b_{8,1}c_{8,3}b_{3,0}^2 + b_{4,0}^2 c_{8,3}b_{3,0}b_{7,1} + b_{4,0}^3 c_{8,3}b_{3,0}^2$

(63) $b_{13,7}^2 + b_{4,0}b_{12,1}b_{3,0}b_{7,1} + b_{4,0}b_{8,1}b_{7,1}^2 + b_{4,0}b_{8,1}b_{14,1} + b_{4,0}b_{8,1}^2 b_{3,0}^2 + b_{4,0}b_{6,1}b_{8,1}^2 + b_{4,0}b_{6,1}^2 b_{3,0}b_{7,0} + b_{4,0}^2 b_{6,1}b_{3,0}^4 + b_{4,0}^2 b_{6,1}b_{12,1} + b_{4,0}^2 b_{6,1}^2 b_{3,0}^2 + b_{4,0}^3 b_{7,0}^2 + b_{4,0}^3 b_{8,1}b_{3,0}^2 + b_{4,0}^4 b_{3,0}b_{7,0} + b_{4,0}^5 b_{3,0}^2 + b_{4,0}^5 b_{6,1} + b_{4,0}c_{8,3}b_{7,1}^2 + b_{4,0}^3 c_{8,3}b_{3,0}^2 + c_{8,3}^2 b_{3,0}b_{7,0} + b_{4,0}b_{6,1}c_{8,3}^2$

(64) $b_{14,1}b_{13,7} + b_{12,7}b_{15,13} + b_{8,1}^2 b_{11,5} + b_{4,0}b_{12,1}b_{11,5} + b_{6,1}b_{8,1}c_{8,3}b_{5,0} + b_{6,1}^2 c_{8,3}b_{7,0} + b_{4,0}c_{8,3}b_{15,13} + b_{4,0}b_{6,1}c_{8,3}b_{3,0}^3 + b_{4,0}b_{6,1}^2 c_{8,3}b_{3,0} + b_{4,0}^2 b_{6,1}c_{8,3}b_{5,0} + b_{4,0}^3 c_{8,3}b_{7,1} + b_{4,0}^3 c_{8,3}b_{7,0} + b_{4,0}c_{8,3}^2 b_{7,0}$

(65) $b_{7,1}^2 b_{13,1} + b_{12,7}b_{15,13} + b_{8,1}b_{12,1}b_{7,1} + b_{4,0}b_{6,1}b_{14,1}b_{3,0} + b_{4,0}b_{6,1}b_{12,1}b_{5,0} + b_{4,0}b_{6,1}b_{8,1}b_{3,0}^3 + b_{4,0}b_{6,1}^2 b_{8,1}b_{3,0} + b_{4,0}b_{6,1}^3 b_{5,0} + b_{4,0}^2 b_{12,1}b_{7,1} + b_{4,0}^2 b_{6,1}^2 b_{7,0} + b_{4,0}^3 b_{15,13} + b_{4,0}^3 b_{12,1}b_{3,0} + b_{4,0}^3 b_{6,1}b_{3,0}^3 + b_{4,0}^3 b_{6,1}^2 b_{3,0} + b_{4,0}^4 b_{6,1}b_{5,0} + b_{4,0}^3 c_{8,3}b_{7,0}$

(66) $b_{14,1}b_{3,0}b_{11,5} + b_{8,1}b_{7,1}b_{13,1} + b_{8,1}^2 b_{12,1} + b_{4,0}b_{11,5}b_{13,1} + b_{4,0}b_{12,1}b_{12,7} + b_{4,0}b_{6,1}b_{12,1}b_{3,0}^2 + b_{4,0}^2 b_{7,1}b_{13,1} + b_{4,0}^3 b_{8,1}b_{3,0}b_{5,0} + b_{4,0}^3 b_{8,1}^2 + b_{4,0}^4 b_{12,1} + b_{4,0}^5 b_{8,1} + c_{8,3}b_{12,1}b_{3,0}b_{5,0} + b_{8,1}c_{8,3}b_{3,0}^4 + b_{6,1}c_{8,3}b_{7,0}^2 + b_{6,1}c_{8,3}b_{14,1} + b_{6,1}b_{8,1}c_{8,3}b_{3,0}^2 + b_{6,1}^2 b_{8,1}c_{8,3} + b_{4,0}b_{8,1}c_{8,3}b_{3,0}b_{5,0} + b_{4,0}b_{8,1}^2 c_{8,3} + b_{4,0}b_{6,1}c_{8,3}b_{3,0}b_{7,0} + b_{4,0}^2 c_{8,3}b_{3,0}^4 + b_{4,0}^2 c_{8,3}b_{12,1} + b_{4,0}^2 b_{6,1}c_{8,3}b_{3,0}^2 + b_{4,0}^2 b_{6,1}^2 c_{8,3} + b_{4,0}^5 c_{8,3}$

(67) $b_{13,1}b_{15,13} + b_{14,1}b_{7,1}^2 + b_{8,1}b_{7,1}b_{13,1} + b_{8,1}^2 b_{12,1} + b_{4,0}b_{6,1}b_{12,1}b_{3,0}^2 + b_{4,0}^2 b_{7,1}b_{13,1} + b_{4,0}^2 b_{12,1}b_{3,0}b_{5,0} + b_{4,0}^2 b_{8,1}b_{3,0}^4 + b_{4,0}^2 b_{6,1}b_{7,0}^2 + b_{4,0}^2 b_{6,1}b_{14,1} + b_{4,0}^2 b_{6,1}b_{8,1}b_{3,0}^2 + :^2_{4,0}\ b_{6,1}^2 b_{8,1} + b_{4,0}^3 b_{8,1}b_{3,0}b_{5,0} + b_{4,0}^3 b_{8,1}^2 + b_{4,0}^3 b_{6,1}b_{3,0}b_{7,0} + b_{4,0}^4 b_{3,0}^4 + b_{4,0}^4 b_{6,1}b_{3,0}^2 + b_{4,0}^4 b_{6,1}^2 + b_{4,0}^7$

(68) $b_{13,7}b_{15,13} + b_{8,1}b_{7,1}b_{13,1} + b_{8,1}^2 b_{12,1} + b_{4,0}b_{6,1}b_{12,1}b_{3,0}^2 + b_{4,0}^2 b_{7,1}b_{13,1} + b_{4,0}^3 b_{8,1}b_{3,0}b_{5,0} + b_{4,0}^3 b_{8,1}^2 + b_{4,0}^4 b_{12,1} + b_{4,0}^5 b_{8,1} + c_{8,3}b_{12,1}b_{3,0}b_{5,0} + b_{6,1}c_{8,3}b_{7,0}^2 + b_{6,1}c_{8,3}b_{14,1} + b_{4,0}b_{8,1}c_{8,3}b_{3,0}b_{5,0} + b_{4,0}b_{8,1}^2 c_{8,3} + b_{4,0}^3 b_{8,1}c_{8,3} + c_{8,3}^2 b_{3,0}^4 + b_{6,1}c_{8,3}^2 b_{3,0}^2 + b_{6,1}^2 c_{8,3}^2 + b_{4,0}^3 c_{8,3}^2$

(69) $b_{7,1}b_{11,5}b_{13,1} + b_{14,1}^2b_{3,0} + b_{8,1}b_{12,1}b_{11,5} + b_{8,1}^3b_{7,0} + b_{6,1}b_{8,1}^2b_{3,0}^3 + b_{6,1}^4b_{7,0} + b_{4,0}b_{12,7}b_{15,13} + b_{4,0}b_{12,1}^2b_{3,0} + b_{4,0}b_{8,1}b_{12,1}b_{7,1} + b_{4,0}b_{8,1}^2b_{11,5} + b_{4,0}b_{6,1}b_{14,1}b_{7,0} + b_{4,0}b_{6,1}b_{8,1}^2b_{5,0} + b_{4,0}b_{6,1}^2b_{8,1}b_{7,0} + b_{4,0}b_{6,1}^3b_{3,0}^3 + b_{4,0}b_{6,1}^4b_{3,0} + b_{4,0}^2b_{8,1}^2b_{7,0} + b_{4,0}^2b_{6,1}b_{14,1}b_{3,0} + b_{4,0}^2b_{6,1}b_{12,1}b_{5,0} + b_{4,0}^2b_{6,1}b_{8,1}b_{3,0}^3 + b_{4,0}^2b_{6,1}^3b_{5,0} + b_{4,0}^3b_{12,1}b_{7,1} + b_{4,0}^3b_{8,1}^2b_{3,0} + b_{4,0}^4b_{15,13} + b_{4,0}^4b_{12,1}b_{3,0} + b_{4,0}^4b_{8,1}b_{7,0} + b_{4,0}^4b_{6,1}b_{3,0}^3 + b_{4,0}^4b_{6,1}^2b_{3,0} + b_{4,0}^6b_{7,1} + b_{8,1}c_{8,3}b_{15,13} + b_{8,1}^2c_{8,3}b_{7,1} + b_{6,1}c_{8,3}b_{14,1}b_{3,0} + b_{6,1}b_{8,1}c_{8,3}b_{3,0}^3 + b_{6,1}^2b_{8,1}c_{8,3}b_{3,0} + b_{6,1}^3c_{8,3}b_{5,0} + b_{4,0}c_{8,3}b_{12,1}b_{7,1} + b_{4,0}b_{6,1}^2c_{8,3}b_{7,0} + b_{4,0}^2c_{8,3}b_{15,13} + b_{4,0}^2b_{6,1}c_{8,3}b_{3,0}^3 + b_{4,0}^2b_{6,1}^2c_{8,3}b_{3,0} + b_{4,0}^3b_{6,1}c_{8,3}b_{5,0} + b_{4,0}^4c_{8,3}b_{7,0} + b_{8,1}c_{8,3}^2b_{7,0} + b_{4,0}^2c_{8,3}^2b_{7,0}$

(70) $b_{8,1}b_{11,5}b_{13,1} + b_{8,1}b_{12,1}b_{12,7} + b_{4,0}b_{14,1}b_{7,0}^2 + b_{4,0}b_{14,1}^2 + b_{4,0}b_{8,1}b_{7,1}b_{13,1} + b_{4,0}^2b_{12,1}^2 + b_{4,0}^3b_{7,1}b_{13,1} + b_{4,0}^4b_{8,1}b_{3,0}b_{5,0} + b_{4,0}^4b_{8,1}^2 + b_{4,0}^6b_{8,1} + c_{8,3}b_{12,7}^2 + b_{4,0}c_{8,3}b_{7,1}b_{13,1} + b_{4,0}^2b_{8,1}c_{8,3}b_{3,0}b_{5,0} + b_{4,0}^2b_{8,1}^2c_{8,3} + b_{4,0}^4b_{8,1}c_{8,3} + b_{8,1}c_{8,3}^2b_{3,0}b_{5,0} + b_{8,1}^2c_{8,3}^2 + b_{4,0}^2b_{8,1}c_{8,3}^2$

(71) $b_{8,1}b_{12,7}b_{13,1} + b_{8,1}b_{12,1}b_{13,7} + b_{4,0}b_{14,1}b_{15,13} + b_{4,0}b_{12,1}b_{14,1}b_{3,0} + b_{4,0}b_{12,1}^2b_{5,0} + b_{4,0}b_{8,1}b_{14,1}b_{7,1} + b_{4,0}^2b_{14,1}b_{11,5} + b_{4,0}^2b_{12,7}b_{13,1} + b_{4,0}^2b_{12,1}b_{13,7} + b_{4,0}^2b_{12,1}b_{13,1} + b_{4,0}^2b_{8,1}^2b_{3,0}^3 + b_{4,0}^2b_{6,1}b_{8,1}^2b_{3,0} + b_{4,0}^2b_{6,1}^2b_{8,1}b_{5,0} + b_{4,0}^3b_{14,1}b_{7,1} + b_{4,0}^3b_{6,1}b_{15,13} + b_{4,0}^3b_{6,1}b_{8,1}b_{7,0} + b_{4,0}^4b_{12,1}b_{5,0} + b_{4,0}^4b_{6,1}^2b_{5,0} + b_{4,0}^5b_{13,1} + b_{4,0}^5b_{8,1}b_{5,0} + b_{4,0}^5b_{6,1}b_{7,0} + b_{4,0}^6b_{3,0}^3 + b_{4,0}^6b_{6,1}b_{3,0} + b_{4,0}^7b_{5,0} + b_{4,0}c_{8,3}b_{14,1}b_{7,1} + b_{4,0}c_{8,3}b_{14,1}b_{7,0} + b_{4,0}b_{8,1}c_{8,3}b_{13,1} + b_{4,0}b_{6,1}c_{8,3}b_{12,1}b_{3,0} + b_{4,0}^3b_{6,1}c_{8,3}b_{7,0}$

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