Computational Homotopy

Graham Ellis

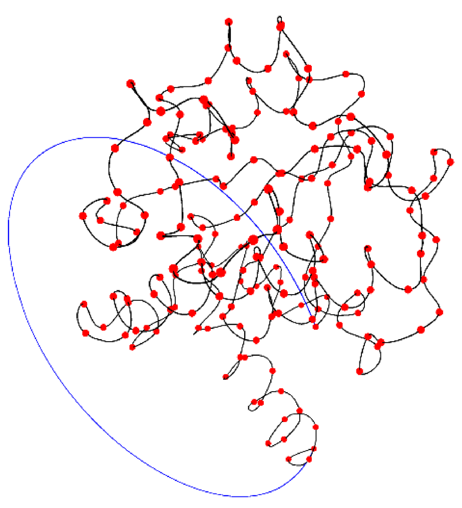
National University of Ireland, Galway

graham.ellis@nuigalway.ie

Abstract

HAP is a software package for basic computations in homotopy theory, especial those related to the fundamental group. It is distributed as part of the GAP system for computational algebra.

1. Fundamental group



Proposition 1.1 *[2] The alpha carbon atoms of the Thermus Thermophilus protein determine a knot K with peripheral system*

$$\pi_1(\partial K) \cong \langle a,b|aba^{-1}b^{-1}\rangle \to \pi_1(\mathbb{R}^3\setminus K) \cong \langle x,y|xyx=yxy\rangle$$
$$a \mapsto x^{-2}yx^2y, \quad b \mapsto x .$$

```
gap> K:=ReadPDBfile("1V2X.pdb");
Pure permutahedral complex of dimension 3

gap> Y:=RegularCWComplex(K);;
Regular CW-complex of dimension 3

gap> i:=Boundary(Y);
Map of regular CW-complexes

gap> phi:=FundamentalGroup(i,22495);
[ f1, f2 ] -> [ f1^-3*f2*f1^2*f2*f1, f1 ]
```

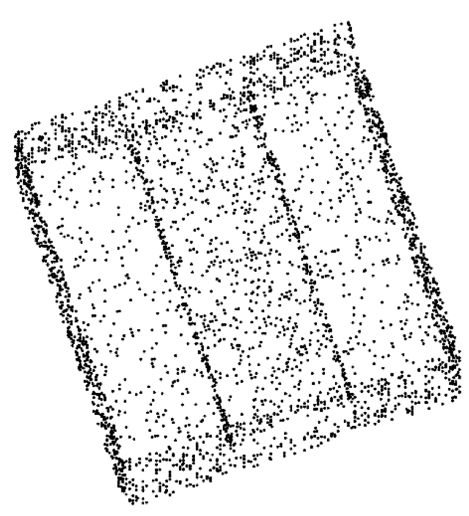
Proposition 1.2 *The fundamental group of Quillen's complex for the symmetric group S_{10} at the prime $p=3$ is a free product*

$$\pi_1(\Delta\mathcal{A}_3(S_{10})) \cong F_{25200} * G$$

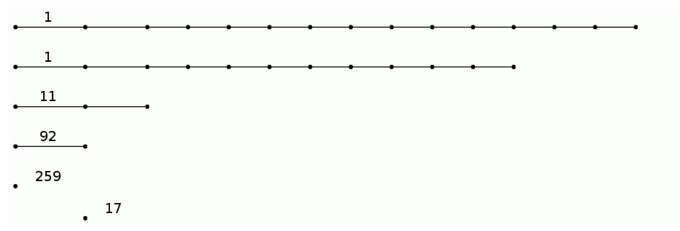
of a free group of rank 25200 with an infinite nonabelian group G defined on 53 generators.

```
gap> G:=SymmetricGroup(10);;
gap> K:=QuillenComplex(G,3);
Simplicial complex of dimension 2

gap> Pi:=FundamentalGroup(K);
<fp group of size infinity with 25253
                          generators>

gap> #etc
```

2. Persistent homology



Proposition 2.1 *[6] The given clould of 3527 points in $\mathbb{R}^3$ determines a filtered cubical complex with the rational β_1 barcode:*




```
gap> T:=Read("euclideanpoints.txt");;
gap> F:=ThickeningFiltration(T,20);
Filtered cubical complex of dimension 3

gap> P:=PersistentHomology(F,1);;
gap> BarCodeCompactDisplay(P);
```

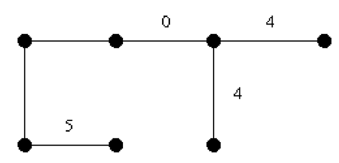
3. Cohomology of finite groups

Theorem 3.1 *[3] The Mathieu simple group M_{24} has fourth integral cohomology*

$$H^4(M_{24},\mathbb{Z}) = \mathbb{Z}_{12} .$$

```
gap> GroupCohomology(MathieuGroup(24),3);
gap> [ 4, 3 ]
```

4. Cohomology of infinite groups



Theorem 4.1 *[7] The Artin group G defined by the given Dynkin diagram has fifth integral cohomology*

$$H^5(G,\mathbb{Z}) = \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} .$$

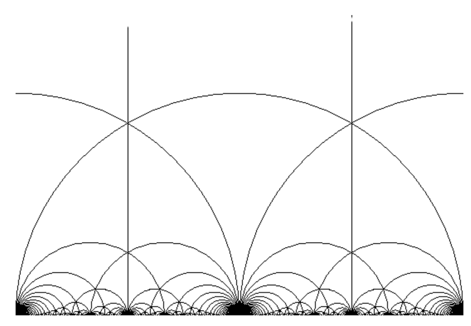
```
gap> D:=[ [1,[2,3],[4,3]], [2,[3,3],[5,0]],
          [3,[4,5]], [5,[6,4],[7,4]] ];;
gap> GroupCohomology(D,5);
[ 2, 2, 0, 0, 0, 0 ]
```

Theorem 4.2 *The second group G in the table of 4-dimensional Euclidean crystallographic groups has 51st integral homology*

$$H_{51}(G,\mathbb{Z}) = (\mathbb{Z}_2)^{16}.$$

```
gap> G:=SpaceGroup(4,2);
SpaceGroupOnRightBBNWZ( 4, 1, 2, 1, 1 )

gap> GroupHomology(G,51);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2,
  2, 2, 2, 2 ]
```

5. Arithmetic



Theorem 5.1 *[1, 4]*

$$H_5(SL_2(\mathbb{Z}[1/46]),\mathbb{Z}) = \mathbb{Z}_4\oplus\mathbb{Z}_8\oplus(\mathbb{Z}_3)^4$$
$$H_5(PSL_4(\mathbb{Z}),\mathbb{Z}) = (\mathbb{Z}_2)^{13}$$
$$H_5(SL_2(\mathcal{O}(\sqrt{-6})),\mathbb{Z}) = \mathbb{Z}_2\oplus\mathbb{Z}_6$$

```
gap> R:=ResolutionSL2Z(46,6);
Resolution of length 6 in characteristic 0

gap> Homology(TensorWithIntegers(R),5);
[ 3, 3, 12, 24 ];

gap> #etc.
```

6. Cohomology rings

Proposition 6.1 *(cf. [5]) The dihedral group of order 512 has mod 2 cohomology ring*

$$H^*(D_{256},\mathbb{Z}_2) = \mathbb{Z}_2[x,y,z]/\langle xy+y^2\rangle$$

with x,y in degree 1 and z in degree 2.

```
gap> Mod2CohomologyRingPresentation(
                       DihedralGroup(512));
Graded algebra GF(2)[ x_1, x_2, x_3 ] /
                          [ x_1*x_2+x_2^2 ]
with indeterminate degrees [ 1, 1, 2 ]
```

7. Homotopy n-types

Proposition 7.1 *The inner-automorphism homomorphism $\iota\colon D_{72}\to Aut(D_{72})$ for the dihedral group of order 144 is a crossed module and, as such, represents a homotopy 2-type X. This is the 55th homotopy 2-type of order 24 and has fifth integral homology*

$$H_5(X,\mathbb{Z}) = (\mathbb{Z}_2)^{10}\oplus\mathbb{Z}_6 .$$

```
gap> C:=AutomorphismGroupAsCatOneGroup(
                         DihedralGroup(72));;

gap> IdQuasiCatOneGroup(C);
[ 24, 55 ]

gap> D:=SmallQuasiCatOneGroup(24,55);;

gap> Homology(D,5);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 6 ]
```

8. Homotopy groups

Proposition 8.1 *Let $K(G,1)$ be an Eilenberg-Mac Lane space for the free nilpotent group G of class 2 on four generators. Let $SK(G,1)$ denote the suspension. Then*

$$\pi_3(SK(G,1)) \cong \mathbb{Z}^{30} .$$

```
gap> F:=FreeGroup(4);;
gap> G:=NilpotentQuotient(F,2);;
gap> ThirdHomotopyGroupOfSuspensionB(G);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0 ]
```

Proposition 8.2 *The affine braid group $G=\langle x,y,z \mid xyx=yxy, xzx=zxz, yzy=zyz\rangle$ admits a 2-dimensionbal classifying space.*

```
gap> F:=FreeGroup(6);;
gap> x:=F.1;;y:=F.2;;z:=F.3;;
gap> a:=F.4;;b:=F.5;;c:=F.6;;
gap> rels:=[a^-1*x*y, b^-1*y*z, c^-1*z*x,
a*x*(y*a)^-1, b*y*(z*b)^-1, c*z*(x*c)^-1];;

gap> IsAspherical(F,rels);
Presentation is aspherical.
```

Proposition 8.3 *The Eilenberg-Mac Lane space $K(\mathbb{Z}/2\mathbb{Z},2)$ admits a CW-structure with $1,0,1,1,2,3,5,8,13,21,\ldots$ cells in dimensions $0,1,2,3,4,5,6,7,8,9,10,\ldots$.*

```
gap> K:=EilenbergMacLaneSimplicialGroup(
                       CyclicGroup(2),2,10);;
gap> CK:=ChainComplexOfSimplicialGroup(K);;
gap> D:=CoreducedChainComplex(CK);;
gap> List([0..9],D!.dimension);
[ 1, 0, 1, 1, 2, 3, 5, 8, 13, 21 ]
```

References

[1] Bui A.T. and G. Ellis. The homology of $Sl_2(Z[1/m])$ for small m. *J. Algebra (in press)*, 2014.

[2] P. Brendel, P. Dłotko, G. Ellis, M. Juda, and M. Mrozek. Computing fundamental groups from point clouds. *preprint*, 2014.

[3] M. Dutour Sikirić and G. Ellis. Wythoff polytopes and low-dimensional homology of Mathieu groups. *J. Algebra*, 322(11):4143–4150, 2009.

[4] M. Dutour Sikirić, G. Ellis, and A. Schürmann. On the integral homology of $\mathrm{PSL}_4(\mathbb{Z})$ and other arithmetic groups. *J. Number Theory*, 131(12):2368–2375, 2011.

[5] G. Ellis, D.J. Green, and S.A. King. The mod-2 cohomology ring of the third Conway group is Cohen-Macaulay. *Algebr. Geom. Topol.*, 11(2):719–734, 2011.

[6] G. Ellis and F. Hegarty. Computational homotopy of finite regular cw-spaces. *Journal of Homotopy and Related Structures*, pages 1–30, 2013.

[7] G. Ellis and E. Sköldberg. The $K(\pi,1)$ conjecture for a class of Artin groups. *Comment. Math. Helv.*, 85(2):409–415, 2010.

Research supported by ESF network ACAT