HOMOTOPY 2-TYPES OF LOW ORDER

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Abstract. There is a well-known equivalence between the homotopy types of connected CW-spaces X with $\pi_n X=0$ for $n \neq 1,2$ and the quasi-isomorphism classes of crossed modules $\partial\colon M \to P$ [16]. When the homotopy groups $\pi_1 X$ and $\pi_2 X$ are finite one can represent the homotopy type of X by a crossed module in which M and P are finite groups. We define the *order* of such a crossed module to be $|\partial| = |M| \times |P|$, and the *order* of a quasi-isomorphism class of crossed modules to be the least order of any crossed module in the class. We then define the *order* of a homotopy 2-type X to be the order of the corresponding quasi-isomorphism class of crossed modules. In this paper we describe a computer implementation that inputs a finite crossed module of reasonably small order and returns a quasi-isomorphic crossed module of least order. Underlying the function is a catalogue of all quasi-isomorphism classes of order $m \leq 127, m \neq 32, 64, 81, 96$ and a catalogue of all isomorphism classes of crossed modules of order $m \leq 255$.

1. Introduction

An important resource for finite group theorists is the computer classification of all groups G of low order. This classification is available in the MAGMA [4] and GAP [11] computer systems and, for example, can be used to: (i) list representatives of all isomorphism classes of groups G of a given order m; (ii) identify the isomorphism class of a user-defined group G in terms of a pair (m,k) where m is the order of G and k is a catalogue number.

In this paper we build on work of Alp and Wensley [1] and develop the beginnings of an analogous resource for homotopy types of connected CW-spaces X with $\pi_n X = 0$ for $n \neq 1,2$. The homotopy type of X is called a *homotopy* 2*-type*. It is well-known that such a homotopy type can be modelled by a group homomorphism $\partial\colon M \to P$ and group action $(p,m) \mapsto {}^p m$ of P on M satisfying

(1) $\partial({}^p m) = p(\partial m)p^{-1}$
(2) ${}^{\partial m} m' = m m' m^{-1}$

for $p \in P$ and $m, m' \in M$. Such a homomorphism and action constitute a *crossed module*. The model is such that $\pi_n X \cong \pi_n(\partial)$ for $n = 1,2$ where one defines $\pi_1(\partial) = P/\operatorname{im}\partial$ and $\pi_2(\partial) = \ker\partial$. A *morphism* of crossed modules $\phi_*\colon (\partial\colon M \to P) \to (\partial'\colon M' \to P')$ consists of two group homomorphisms $\phi_1\colon P \to P'$, $\phi_2\colon M \to M'$ that satisfy $\partial'\phi_2(m) = \phi_1\partial(m)$, $\phi_2({}^p m) = {}^{\phi_1 p}\phi_2(m)$ for $m, m' \in M, p \in P$. A morphism induces canonical homomorphisms $\pi_n(\phi_*)\colon \pi_n(\partial) \to \pi_n(\partial')$ for $n = 1,2$. The morphism ϕ_* is said to be an *isomorphism* if ϕ_n is an isomorphism for $n = 1,2$. The morphism ϕ_* is said to be a *quasi-isomorphism* if $\pi_n(\phi_*)$

Key words and phrases. crossed module, homotopy 2-type, quasi-isomorphism.

The first author thanks the IST, Austria for its hospitality during the writing of this paper and the European Science Foundation network on *Applied and Computational Algebraic Topology* for support. The second author has been supported by an NUI Galway, College of Science, PhD fellowship. We thank the referee for suggesting several improvements to the paper.

is an isomorphism for $n = 1, 2$. We write $\partial \overset{\simeq}{\leftrightarrow} \partial'$ to denote the existence of either a quasi-isomorphism $\partial \to \partial'$ or a quasi-isomorphism $\partial' \to \partial$. Two crossed modules ∂, ∂' are said to be *quasi-isomorphic* if there exists a sequence of quasi-isomorphisms $\partial = \partial_1 \overset{\simeq}{\leftrightarrow} \partial_2 \overset{\simeq}{\leftrightarrow} \partial_3 \overset{\simeq}{\leftrightarrow} \cdots \overset{\simeq}{\leftrightarrow} \partial_\ell = \partial'$ of arbitrary length $\ell - 1$. We write $\partial \simeq \partial'$ to denote that ∂ is quasi-isomorphic to ∂'. Note that $\simeq$ is an equivalence relation on crossed modules; the corresponding equivalence classes are called *quasi-isomorphism classes*. We emphasize that two crossed modules ∂, ∂' may be quasi-isomorphic without the existence of any quasi-isomorphism directly between ∂ and ∂'.

Mac Lane and Whitehead [16] showed that there is a one-one correspondence between homotopy 2-types and quasi-isomorphism classes of crossed modules (see also [15]). We define the *order* of a crossed module $\partial\colon M \to P$ to be the product $|\partial| = |M| \times |P|$ of the orders of the groups M, P. We define the *order* of a quasi-isomorphism class of crossed modules to be the least order of any crossed module in the class. We define the *order* of a homotopy 2-type X to be the order of the corresponding quasi-isomorphism class of crossed modules. A homotopy 2-type X can also be represented by the fundamental group $\pi_1 X$, the $\pi_1 X$-module $\pi_2 X$ and a cohomology class $\kappa \in H^3(\pi_1 X, \pi_2 X)$ known as the *Postnikov invariant*. The Postnikov invariant κ is the trivial cohomology class if and only if the homotopy 2-type can be represented by a crossed module $\partial\colon M \to P$ with $\partial = 0$. In this case we deem the homotopy 2-type, and also the quasi-isomorphism type, to be *trivial*.

In this paper we describe two computer functions, both of which have been implemented by the second author in the HAP package [9] for the computer algebra system GAP [11]. The first function lists representatives for all the quasi-isomorphism classes of crossed modules of a given order $m \leq 127$, $m \neq 32, 64, 81, 96$. The second function inputs a user-defined crossed module (of order possibly greater than 127) and tries to return numbers (m, k) that identify the least order m of any quasi-isomorphic crossed module and a catalogue number k that uniquely identifies the quasi-isomorphism class of the input. The latter certainly succeeds if the input is of order $\leq 127, \neq 32, 64, 81, 96$. We have used the implementation of these two functions, and related functions, to compile Table 1. The table uses the notation:

$I2(m)$	$=$	number of isomorphism classes of crossed modules of order m.
$Q2(m)$	$=$	number of homotopy 2-types of order m
	$=$	number of quasi-isomorphism classes of order m.
$T2(m)$	$=$	number of trivial homotopy 2-types of order m.

It is an easy exercise to see that $I2(p) = Q2(p) = T2(p) = 2$ for p a prime and so we omit prime values of m from the table. It is also easy to show that for primes $p < q$ we have $I2(pq) = Q2(pq) = T2(pq) = 6$ when p divides $q - 1$ and $I2(pq) = Q2(pq) = T2(pq) = 4$ when p does not divide $q - 1$ and so these values of m are also omitted from the table. (To establish the formulae one uses that: the cyclic group of order p can act non-trivially on the cyclic group of order q precisely when p divides $q - 1$; the only groups of order p or order pq with p not dividing $q - 1$ are the cyclic groups; the only groups of order pq with p dividing $q - 1$ are the cyclic group and one non-abelian semi-direct product of cyclic groups.) The table suggests the general formulae:

$I2(m) - 1 = Q2(m) = T2(m) = 5$ for $m = p^2$;
$I2(m) - 4 = Q2(m) = T2(m) = 14$ for $m = p^3, p \geq 3$;
$I2(m) - 2 = Q2(m) = T2(m) = 18$ for $m = 4p, p \geq 5, p \equiv 1 \bmod 4$;
$I2(m) - 2 = Q2(m) = T2(m) = 16$ for $m = 4p, p \geq 5, p \equiv 3 \bmod 4$;

and more complicated formulae for the cases $m = q^2p, q \geq 3$ and $m = pqr$ with p, q, r distinct primes. These formulae can be verified for a good range of m using the above mentioned computer functions.

m	1	4	8	9	12	16	18	20	24	25	27	28	30	32	36	40	42
$I2(m)$	1	6	18	6	20	62	22	20	73	6	18	18	20	251	78	72	26
$Q2(m)$	1	5	14	5	18	43	19	18	61	5	14	16	20	A	63	60	26
$T2(m)$	1	5	14	5	18	42	19	18	61	5	14	16	20	152	63	60	26

m	44	45	48	49	50	52	54	56	60	63	64	66	68	70	72
$I2(m)$	18	12	296	6	22	20	81	68	77	18	1276	20	20	20	325
$Q2(m)$	16	10	224	5	19	18	65	56	73	16	B	20	18	20	251
$T2(m)$	16	10	220	5	19	18	65	56	73	16	697	20	18	20	251

m	75	76	78	80	81	84	88	90	92	96	98	99	100	102	104
$I2(m)$	14	18	26	302	64	90	66	76	18	1446	22	12	87	20	72
$Q2(m)$	12	16	26	230	C	84	54	66	16	D	19	10	71	20	60
$T2(m)$	12	16	26	226	44	84	54	66	16	971	19	10	71	20	60

m	105	108	110	112	114	116	117	120	121	124	125	126	128
$I2(m)$	12	308	26	270	26	20	18	342	6	18	18	102	9120
$Q2(m)$	12	238	26	202	26	18	16	302	5	16	14	92	?
$T2(m)$	12	238	26	198	26	18	16	302	5	16	14	92	4680

$$158 \leq A \leq 171,\ 727 \leq B \leq 831,\ 45 \leq C \leq 46,\ 996 \leq D \leq 1052$$

TABLE 1

Perhaps not surprisingly, the table shows that most of the homotopy 2-types of low order have trivial Postnikov invariant. It shows that the smallest homotopy 2-type with non-trivial Postnikov invariant has order 16, and that there is just one non-trivial homotopy 2-type of this order. A straightforward computer analysis shows that this homotopy type is represented by the crossed module with $M = \langle x \,|\, x^4 = 1\rangle$, $P = \langle a \,|\, a^4 = 1\rangle$, ${}^ax = x^3$, $\partial(x) = a^2$. It is also represented by the crossed module $M = \langle x, y \,|\, x^2 = y^2 = [x, y] = 1\rangle$, $P = \langle a \,|\, a^4 = 1\rangle$, ${}^ax = xy$, ${}^ay = y$, $\partial(x) = a^2$, $\partial(y) = 1$. No other crossed module of order 16 represents the unique smallest homotopy 2-type with non-trivial Postnikov invariant. We remark that it has been observed previously that the second of the crossed modules representing this homotopy type corresponds to a non-trivial Postnikov invariant; see for instance the example of Section 7 in [12] and Example 12.7.12 in [5].

The paper contains no formal theorem or formal proof. Our main result, the complete and irredundant list of integral homotopy 2-types of order $m \leq 127, m \neq 32, 64, 81, 96$ and a complete but probably redundant list for $m = 32, 64, 81, 96$, is presented in the form of a collection of implemented GAP functions. These functions have been made publicly available as part of the HAP package [9] for the GAP system. The user interface to the functions is described in detail in Section 2. The homotopy types are exhibited in the form of finite crossed modules and should provide a useful bank of examples to complement the growing literature on finite crossed modules (*cf.* [6, 14, 18, 3, 19, 17]) and substantial literature on crossed modules.

Our proof of the main result is explained informally in Sections 3-5. It is based on the classical result of MacLane and Whitehead [16] on crossed modules mentioned above, but highlights the theoretical and computational advantages of working with the isomorphic notion of a *cat*^{1}*-group*. It involves computations of the low-dimensional integral homology of classifying spaces of crossed modules, and computations of third twisted cohomology of finite groups. The motivated reader should be able to reproduce our catalogue of homotopy 2-types using the explanation in Sections 3-5 together with the algorithm for twisted homology of groups given in [8] and the algorithm for homology of classifying spaces of crossed modules given in [10].

The values of $I2(m)$ for $m \leq 63$ are available from the software [2] described in [1]. The values of $I2(m)$ for higher m are obtained from a function for listing non-isomorphic crossed modules of given order which was designed and implemented in [9] by the second author. Details are given in Section 3. We are grateful to Alexander Hulpke for providing a key step in the implementation of our algorithm for testing isomorphism of two crossed modules.

For each of the $I2(m)$ crossed modules of order m we apply a refinement of an algorithm in our previous paper [10] that attempts to find a smaller quasi-isomorphic crossed module. In this way we obtain an upper bound for $Q2(m)$. Details are given in Section 4.

To prove that the upper bound equals $Q2(m)$ we need a method for establishing that two crossed modules ∂, ∂' are not quasi-isomorphic. We do this by computing the quasi-isomorphism invariants $\pi_1(\partial)$, $\pi_2(\partial)$, $H^3(\pi_1(\partial), \pi_2(\partial))$ and $H_n(X, \mathbb{Z})$. The last invariant is the homology of the homotopy 2-type X represented by ∂ and is computed using our algorithm described in [10]. We use the group cohomology routines in our HAP package [9] to compute $H^3(\pi_1(\partial), \pi_2(\partial))$. Details are given in Section 5.

2. Computer implementation

It is well-known that the notion of a crossed module can be reformulated as a "group with compatible category structure". We use such a reformulation both for implementing algorithms and for checking correctness of algorithms. There are several variants of the reformulation and we opt to work with the following notion due to J-L. Loday [15].

A *cat*^{1}*-group* consists of a pair of group endomorphisms $s, t\colon G \to G$ satisfying $ts = s$, $st = t$ and $[\ker s, \ker t] = 1$. A *morphism* of cat^1-groups $\phi\colon (G, s, t) \to (G', s', t')$ consists of a group homomorphism $\phi\colon G \to G'$ satisfying $\phi s = s'\phi$ and $\phi t = t'\phi$. A cat^1-group gives rise to a crossed module by taking $M = \ker s$, $P = \operatorname{im} s$ and taking ∂ to be the restriction of t to $\ker s$. Conversely, a crossed module gives rise to a cat^1-group by using the action of P on M to form the semi-direct product $G = M \rtimes P$ and defining the endomorphisms $s, t\colon M \rtimes P \to M \rtimes P$ as $s(m, p) = (1, p)$, $t(m, p) = (1, (\partial m)p)$. It is observed in [15] that these two constructions provide an isomorphism between the category of crossed modules and the category of cat^1-groups. It is thus routine to translate notions of order, homotopy group, quasi-isomorphism and (trivial) quasi-isomorphism class of crossed modules to equivalent notions for cat^1-groups $C = (G, s, t)$. For example: $\pi_1(C) = \operatorname{im}(s)/t(\ker s)$; $\pi_2(C) = \ker s \cap \ker t$; C represents a trivial homotopy 2-type precisely when it is quasi-isomorphic to one in which $s = t$; and so on. We leave details to the reader and use these equivalent notions throughout the remainder of the paper.

We provide four short GAP sessions to illustrate the functionality of our computer implementation of homotopy 2-types available in the HAP package [9]. The first session begins by setting G equal to the 500th group of order 2000 from the database of small groups. It then computes a list L of all possible non-isomorphic cat^1-group structures on G. The list L has length 16.

```
gap> G:=SmallGroup(2000,500);; L:=CatOneGroupsByGroup(G);; Length(L);
16
```

Our second GAP session involves the homomorphism $\alpha_H\colon H \to \mathrm{Aut}(H), h \mapsto \iota_h$ from a group H to its automorphism group which sends $h \in H$ to the inner automorphism $\iota_h\colon H \to H, x \mapsto hxh^{-1}$. The homomorphism α_H is a crossed module with respect to the obvious action of $\mathrm{Aut}(H)$ on H. The session begins by constructing the associated cat^1-group C for H equal to the dihedral group of order 12. The second command in the session determines that the underlying group of C is the 154th group of order 144 in the small groups database, and that C is endowed with the 8th cat^1-structure on this group. The final command identifies the quasi-isomorphism class of C to be the 4th quasi-isomorphism class of order 4.

```
gap> C:=AutomorphismGroupAsCatOneGroup(DihedralGroup(12));;
gap> IdCatOneGroup(C);
[ 144, 154, 8 ]
gap> IdQuasiCatOneGroup(C);
[ 4, 4 ]
```

Our third GAP session begins by constructing the cat^1-group corresponding to the 2nd homotopy 2-type X of order 30. It then uses the algorithm from [10] to compute $H_5(X,\mathbb{Z}) = \mathbb{Z}_{10}$.

```
gap> C:=SmallQuasiCatOneGroup(30,2);; Homology(C,5);
[ 10 ]
```

Our fourth GAP session concerns the crossed module $\partial\colon G \otimes G \to G, g \otimes h \mapsto ghg^{-1}h^{-1}$ involving the nonabelian tensor square of groups introduced by Brown and Loday [7]. The group $G \otimes G$ is generated by symbols $g \otimes h$ for $g, h \in G$ subject to the relations

$$gg' \otimes h = {}^{g}(g' \otimes h)\,(g \otimes h),$$

$$g \otimes hh' = (g \otimes h)\,{}^{h}(g \otimes h'),$$

for $g, g', h, h' \in G$ where the action is defined by

$${}^{x}(g \otimes h) = xgx^{-1} \otimes xhx^{-1}.$$

It is verified in [7] that this structure is a crossed module. Our fourth GAP session first shows that the corresponding cat^1-group C has order 20736 for G equal to the dihedral group of order 72. It then obtains a quasi-isomorphic cat^1-group of order 256. The session ends by computing the homotopy group orders $|\pi_1(C)| = 4$ and $|\pi_1(C)| = 16$. These orders imply that any cat^1-group which is quasi-isomorphic to C must be of order at least $64 = 4 \times 16$.

```
gap> C:=NonabelianTensorSquareAsCatOneGroup(DihedralGroup(72));; Size(C);
20736
gap> D:=QuasiIsomorph(C);; Size(D);
256
gap> Size(HomotopyGroup(D,1)); Size(HomotopyGroup(D,2));
4
16
```

The HAP implementation contains functions for converting a crossed module to a cat^1-group and vice-versa. Each of the above GAP sessions could thus equally well have been performed using equivalent crossed modules.

3. Enumeration of isomorphism classes

The GAP package [2] of Alp and Wensley provides a list of all non-isomorphic cat^1-groups and crossed modules of order ≤ 63. To handle larger examples the second author implemented a function which inputs a finite group G and outputs a list of all non-isomorphic cat^1-group structures (G, s, t). This implementation uses GAP's function `IdGroup(H)` for identifying certain subgroups $H \leq G$ by their order m and catalogue number k and thus works only in cases for which `IdGroup(H)` is implemented.

The algorithm begins by computing a list $\mathbb{L}$ of all normal subgroups N in G and a list $\mathbb{L}'$ of subgroups K in G representing all subgroup conjugacy classes. There are then two steps to the algorithm.

Step 1. For each $N \in \mathbb{L}$ we find all $K \in \mathbb{L}'$ satisfying

- K is isomorphic to G/N. (Here we just test if $\texttt{IdGroup}(K) = \texttt{IdGroup}(G/N)$).
- $|p(K)| = |G/N|$, where $p\colon G \to G/N$ is the quotient homomorphism.

For each such pair N, K the homomorphism p restricts to an isomorphism $p|_K\colon K \to G/N$. We form the inverse isomorphism $(p|_K)^{-1}\colon G/N \to K$ and set $\sigma = (p|_K)^{-1}p\colon G \to G$. By construction we have $\ker \sigma = N$, $\operatorname{im} \sigma = K$ and $\sigma^2 = \sigma$. For each normal subgroup N we compute the list $\mathbb{L}_N$ of such homomorphisms σ.

Step 2. For each pair of normal subgroups N, M in G satisfying $[N, M] = 1$ we consider all $s \in \mathbb{L}_N$, $t \in \mathbb{L}_M$. If $\operatorname{im} s = \operatorname{im} t$ we add the data (G, s, t) to our list of cat^1-group structures on G.

In this manner, all possible cat^1-group structures on G are produced, though isomorphic copies may have been produced by the algorithm.

To test if two cat^1-group structures on a group G are isomorphic we need to access the automorphism group $\operatorname{Aut}(G)$ of the group G. As this automorphism group can be large we follow a suggestion of Alexander Hulpke and use:

(i) the action ${}^fK = f(K)$ of $f \in \operatorname{Aut}(G)$ on subgroups $K \leq G$;

(ii) the action ${}^fs(x) = fs(x)f^{-1}$ of $f \in \operatorname{Aut}(G)$ on endomorphisms $s\colon G \to G$.

For each action we have adapted a GAP implementation of an orbit-stabilizer algorithm written by Alexander Hulpke and used it to compute the orbit of an element under the action and to compute the stabilizer subgroup of this element. A description of the orbit-stabilizer algorithm can be found in [13].

To test if two cat^1-group structures (G, s, t) and (G, s', t') are isomorphic we perform the following steps.

Step 1. We first use GAP's `IdGroup()` function to check that $\operatorname{im} s \cong \operatorname{im} s'$ and $\ker s \cong \ker s'$ and $\ker t \cong \ker t'$. If this check fails then the two cat^1-groups are not isomorphic and we return false.

Step 2. Otherwise we compute the orbit of $\ker s$ under the action of $\mathrm{Aut}(G)$. If $\ker s'$ is not in this orbit then the two cat^1-groups are not isomorphic and we return false. Otherwise we can find an element $f \in \mathrm{Aut}(G)$ such that $\ker s' = {}^f(\ker s)$. We then define $s'' = {}^{f^{-1}}s'$, $t'' = {}^{f^{-1}}t'$ to obtain a cat^1-group (G, s'', t'') which is isomorphic to (G, s', t') and which has the property that $\ker s'' = \ker s$. For ease of notation we redefine $s' := s''$, $t' := t''$. In other words, we replace (G, s', t') by an isomorphic cat^1-group satisfying $\ker s' = \ker s$.

Step 3. We compute the stabilizer subgroup $\mathrm{Stab}(\ker s) \le \mathrm{Aut}(G)$ and the orbit of $\operatorname{im} s$ under the action of $\mathrm{Stab}(\ker s)$. If $\operatorname{im} s'$ is not in this orbit then the two cat^1-groups are not isomorphic and we return false. Otherwise we can find an element $f \in \mathrm{Stab}(\ker s)$ such that $\operatorname{im} s' = {}^f(\operatorname{im} s)$ and then replace (G, s', t') by an isomorphic cat^1-group satisfying $\operatorname{im} s' = \operatorname{im} s$ and $\ker s' = \ker s$.

Step 4. We compute the stabilizer subgroup $\mathrm{Stab}(\operatorname{im} s) \le \mathrm{Stab}(\ker s)$ and the orbit of $\ker t$ under the action of $\mathrm{Stab}(\operatorname{im} s)$. If $\ker t'$ is not in this orbit then the two cat^1-groups are not isomorphic and we return false. Otherwise we replace (G, s', t') by an isomorphic cat^1-group satisfying $\ker t' = \ker t$, $\operatorname{im} s' = \operatorname{im} s$ and $\ker s' = \ker s$.

Step 5. We compute the stabilizer $\mathrm{Stab}(\ker t) \le \mathrm{Stab}(\operatorname{im} s)$ and the orbit of s under the action of $\mathrm{Stab}(\ker t)$. If s' is not in this orbit the two cat^1-groups are not isomorphic and we return false. Otherwise we replace (G, s', t') by an isomorphic cat^1-group satisfying $\ker t' = \ker t$, $s' = s$.

Step 6. We compute the stabilizer $\mathrm{Stab}(s) \le \mathrm{Stab}(\ker t)$ and the orbit of t under the action of $\mathrm{Stab}(s)$. If t' is not in this orbit then the two cat^1-groups are not isomorphic and we return false. Otherwise we return true.

4. Construction of small quasi-isomorphic representatives

Given a cat^1-group (G, s, t) we attempt to find a smaller quasi-isomorphic cat^1-group using (a new implementation of) the following procedure which was described in [10].

Step 1: Given a finite cat^1-group (G, s, t) we search through the normal subgroups of $\ker s$ to find a largest normal subgroup K in $\ker s$ such that K is normal in G and $K \cap \ker t = 1$. We can then choose some generating set X_K for K and construct the group N generated by the set $X_K \cup \{t(x) : x \in X_K\}$. Then N is a normal subgroup of G for which the quotient homomorphism $G \to G/N$ induces a quasi-isomorphism $(G, s, t) \to (G/N, s', t')$.
Step 2: Given a finite cat^1-group (G, s, t) we search through the subgroups of G to find a smallest group $K \subset G$ such that the inclusion induces a quasi-isomorphism $(K, s', t') \hookrightarrow (G, s, t)$. Since we are dealing with finite groups we can test if this inclusion is a quasi-isomorphism simply by checking if the induced homomorphisms $\pi_n K \to \pi_n G$ are surjections and $|\pi_n K| = |\pi_n G|$ for $n = 1, 2$.
Step 3: We repeatedly apply Step 1 followed by Step 2 until no more size reduction is achieved.

To give some feel for the performance of the above QuasiIsomorph procedure we applied it to the 62 cat^1-groups C of order 16 to find:

(a) 42 satisfy $|C| = |\pi_1(C)| \times |\pi_2(C)|$, and so are already unique minimal size representatives of quasi-isomorphism classes and there is nothing to do;

(b) 17 reduce by the procedure to a smaller quasi-isomorphic cat^1-group D satisfying $|D| = |\pi_1(D)| \times |\pi_2(D)|$, so a unique minimal size representative of the class of C is determined;

(c) the remaining three cat^1-groups C, with catalogue numbers [16,2,4], [16,3,4] and [16,4,3], fail to be reduced by the procedure even though all three have $\pi_1(C) = \pi_2(C) = C_2$.

5. Establishing distinct quasi-isomorphism classes

A table of all isomorphism types of cat^1-groups of order at most 255 has been computed and stored in [9]. This table underlies the function `IdCatOneGroup(G)` which returns a triple (m, k_1, k_2) for any cat^1-group G of order $m \le 255$; the integer k_1 is the number of the underlying group of G in the small groups database; the integer k_2 is the number of the cat^1-group structure on G in our table of small cat^1-groups.

The table of isomorphism types of cat^1-groups immediately yields the upper bound $I2(m) \geq Q2(m)$ on the number of quasi-isomorphism types of cat^1-groups. By applying the procedure of Section 4 to each isomorphism type G of order m in the table, and then discarding G if the procedure succeeds in finding a smaller cat^1-group quasi-isomorphic to G, we obtain a list L of (not necessarily minimal size) representatives of quasi-isomorphism types of cat^1-groups of order m. It could be that some pair $G, G' \in L$ are quasi-isomorphic. For each $G \in L$ we have computed the following quasi-isomorphism invariants:

i) the small groups database identifier for $\pi_1(G)$;

ii) the small groups database identifier for $\pi_2(G)$;

iii) the small groups database identifier for the semi-direct product $\pi_2(G) \rtimes \pi_1(G)$ involving the action of the first homotopy group on the second homotopy group;

iv) the abelian invariants of the integral homology group $H_n(X, \mathbb{Z})$ for $n \le 5$ where X is the homotopy 2-type represented by G (the homology algorithm from [10] was used for this).

If for every pair of non-isomorphic cat^1-groups $G, G' \in L$ at least one of the invariants (i)-(iv) yields distinct values we conclude that our list L contains precisely one representative for each quasi-isomorphism class of order m.

In cases where invariants (i)-(iv) are identical for two cat^1-groups $G, G' \in L$ the order of the cohomology group $H^3(\pi_1(G), \pi_2(G))$ was computed using group cohomology functions in [9]. By Mac Lane and Whitehead's result [16], the order of this cohomology group provides an upper bound on the number of quasi-isomorphism types with given fundamental group $\pi_1(G)$ and given second homotopy group $\pi_2(G)$. In some cases this upper bound is sufficient to conclude that G and G' are quasi-isomorphic.

To illustrate this strategy let $C[m, k, d]$ denote the dth cat^1-group structure whose underlying group is the kth group of order m. We shall prove that $C[16, 2, 4]$ is quasi-isomorphic to $C[4, 2, 2]$. Firstly, to each cat^1-group C we associate the trivial crossed module $0\colon \pi_2(C) \to \pi_1(C)$ and denote the corresponding trivial cat^1-group by $\Pi(C)$. We refer to $\Pi(C)$ as the *homotopy cat^1-group* of C. Quasi-isomorphic cat^1-groups $C \simeq C'$ have isomorphic homotopy cat^1-groups $\Pi(C) \cong \Pi(C')$. Computer calculations show that

$$\Pi(C[16, 2, 4]) \cong \Pi(C[16, 3, 4]) \cong \Pi(C[16, 4, 3]) \cong C[4, 2, 2]$$

and that

$$H^3(\pi_1(C[4, 2, 2]), \pi_2(C[4, 2, 2])) = \mathbb{Z}_2. \tag{1}$$

In light of Mac Lane and Whitehead's theory summarized in the Introduction, (1) implies that the four cat^1-groups $C[16,2,4]$, $C[16,3,4]$, $C[16,4,3]$ and $C[4,2,2]$ represent at most two quasi-isomorphism classes. A computer calculation establishes that $C[4,2,2]$ has integral homology different to that of $C[16,3,4]$. A further computation establishes that the integral homology of $C[16,2,4]$ is distinct from that of $C[16,3,4]$. We conclude that $C[16,2,4]$ is quasi-isomorphic to $C[4,2,2]$. A drawback to this method of argument is that it gives no indication of a sequence of quasi-isomorpisms relating $C[16,2,4]$ and $C[4,2,2]$. We know of no algorithm for finding such a sequence.

Using the above strategy a non-redundant list of representatives of all quasi-isomorphism types of order m has been computed and recorded in the software [9] for all $m \leq 127$ excluding $m = 32, 64, 81, 96$. This record is used to implement the function `IdQuasiCatOneGroup(G)` which inputs a cat^1-group G and tries to return the pair of integers (m,k) with m the order of the smallest cat^1-group in the quasi-isomorphism class of G and k a number uniquely identifying this smallest representative. The function first attempts to find a quasi-isomorphism representative of G of order $\leq 127, \neq 32, 64, 81, 96$; if it succeeds it then uses the stored record to produce m and k.

For $m = 32, 64, 81, 96$ the above strategy produces the bounds $158 \leq Q(32) \leq 171$, $727 \leq Q2(64) \leq 831$, $45 \leq Q2(81) \leq 46$ and $996 \leq Q2(96) \leq 1052$.

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