A colimit of classifying spaces

Graham Ellis
Mathematics Department, National University of Ireland, Galway

Roman Mikhailov
Steklov Mathematical Institute, Moscow

June 6, 2008

Abstract

We recall a description of the first non-vanishing homotopy group of a certain (n+1)-ad of spaces and show how it yields group-theoretic formulae for homotopy and homology groups of several specific spaces.

1 Introduction

In this article we give group-theoretic formulae for the nth and $(n+1)$st homotopy groups of the homotopy colimit X of a certain n-cubical diagram of classifying spaces of discrete groups. The formula for $\pi_n X$ is an intersection of normal subgroups; the formula for $\pi_{n+1} X$ involves a finitely presented group which generalizes the nonabelian tensor product of groups introduced in [11, 8]. In some cases the formulae are computational. However, our main motivation is the search for new group-theoretic constructions that can be used to translate homotopy theoretic calculations into the language of combinatorial group theory; the article is thus a continuation of ideas pursued by the second author and H.-J. Baues in [5].

A range of homotopy and homology groups have been described as intersections of subgroups (see for example [3, 5, 6, 10, 12, 18, 20, 25]). Some years ago the first author discovered a Hopf type formula for the higher integral homology of a group [15] using F. Keune's theory of relative nonabelian derived functors [21]. He subsequently observed that the formula could also be obtained as an easy consequence of a higher homotopy van Kampen theorem of R. Brown and J.-L. Loday [8] and published this topological proof with R. Brown in [7]. In this article we develop the methods underlying that topological proof.

Let G be a group with normal subgroups $N_1, \ldots, N_n$. We are interested in the nth and $(n+1)$st homotopy groups of the topological space X arising as the homotopy colimit of classifying spaces $B(G/\prod_{i\in I} N_i)$ where I ranges over all strict subsets $I \subsetneq \{1, \ldots, n\}$. We need a connectivity condition and define an m-tuple of normal subgroups $(N_1, \ldots, N_m)$ to be *connected* if either $m \le 2$ or $m \ge 3$ and for all subsets $I, J \subset \{1, \cdots, m\}$ with $|I| \ge 2, |J| \ge 1$ the following equality holds:

$$\left(\bigcap_{i\in I} N_i\right)\left(\prod_{j\in J} N_j\right) = \bigcap_{i\in I}\left(N_i(\prod_{j\in J} N_j)\right) . \qquad (1)$$

Part of the statement of our main result involves a group $T(N_1, \ldots, N_n)$ which is recalled from [17] in Theorem 12 below. Our main result is:

Theorem 1 *If the $(n-1)$-tuple $(N_1, \ldots, \hat{N}_i, \ldots, N_n)$ is connected for each $1 \le i \le n$ then the above colimit X has*

$$\pi_n(X) \cong \frac{N_1 \cap \ldots \cap N_n}{\prod_{I \cup J = \{1,\ldots,n\}, I \cap J = \emptyset} [\cap_{i \in I} N_i, \cap_{j \in J} N_j]}, \tag{2}$$

$$\pi_{n+1}(X) \cong \ker(\partial \colon T(N_1, \ldots, N_n) \to G). \tag{3}$$

For $n = 2$ isomorphism (2) is given in [6] and reads

$$\pi_2(X) \cong \frac{N_1 \cap N_2}{[N_1, N_2]}. \tag{4}$$

For $n = 2$ isomorphism (3) is given in [8]. For $n = 3$ isomorphism (2) is seemingly new and reads

$$\pi_3(X) = \frac{N_1 \cap N_2 \cap N_3}{[N_1, N_2 \cap N_3][N_2, N_1 \cap N_3][N_3, N_1 \cap N_2]}.$$

We shall explain how isomorphism (2) implies: the group-theoretic description of $\pi_n(S^2)$ given by J. Wu [25]; homomorphisms of second and third homotopy groups given in [20] and [5] respectively; Hopf type formulae for the higher homology of a group given in [15, 7]; seemingly new results on the existence of torsion in certain groups arising from "almost aspherical" presentations. We also indicate how isomorphism (3) yields explicit calculations of homotopy groups such as $\pi_{n+2}(S^n K(G,1))$ for a range of groups G.

Our approach is heavily influenced by work of R. Brown and J.-L. Loday. In [22] Loday introduced a functor Π: (n-cubes of spaces) $\to$ (cat^n-groups) from topology to algebra. Brown and Loday [8] proved that this functor preserves certain connectivity conditions and certain colimits. An explicit description of colimits in the algebraic category can be found in [6] for $n = 1$, in [22, 8] for $n = 2$, and in [17] for the general case $n \ge 1$. We shall explain how Theorem 1 is a fairly immediate consequence of this body of work. Indeed, Theorem 1 is really just an observation, but as this observation seems not to have been made before, and as it has a number of useful consequences, we give it the status of a theorem.

We begin with the case of three normal subgroups.

2 The case $n = 3$

Suppose that L, M, N are normal subgroups of a group G. This data gives rise to a commutative cube of spaces

$$\begin{tikzcd}
 & B(G) \arrow[rr] \arrow[dl] \arrow[dd] & & B(G/L) \arrow[dl] \arrow[dd] \\
B(G/M) \arrow[rr] \arrow[dd] & & B(G/LM) \arrow[dd] & \\
 & B(G/N) \arrow[rr] \arrow[dl] & & B(G/LN) \arrow[dl] \\
B(G/MN) \arrow[rr] & & X &
\end{tikzcd}$$

in which $B(G)$ denotes the classifying space of G and X is the homotopy pushout of the diagram of classifying spaces. Working up to homotopy type, we can extend this cube of spaces to a diagram of 27 spaces (some of which are shown in Figure 1)

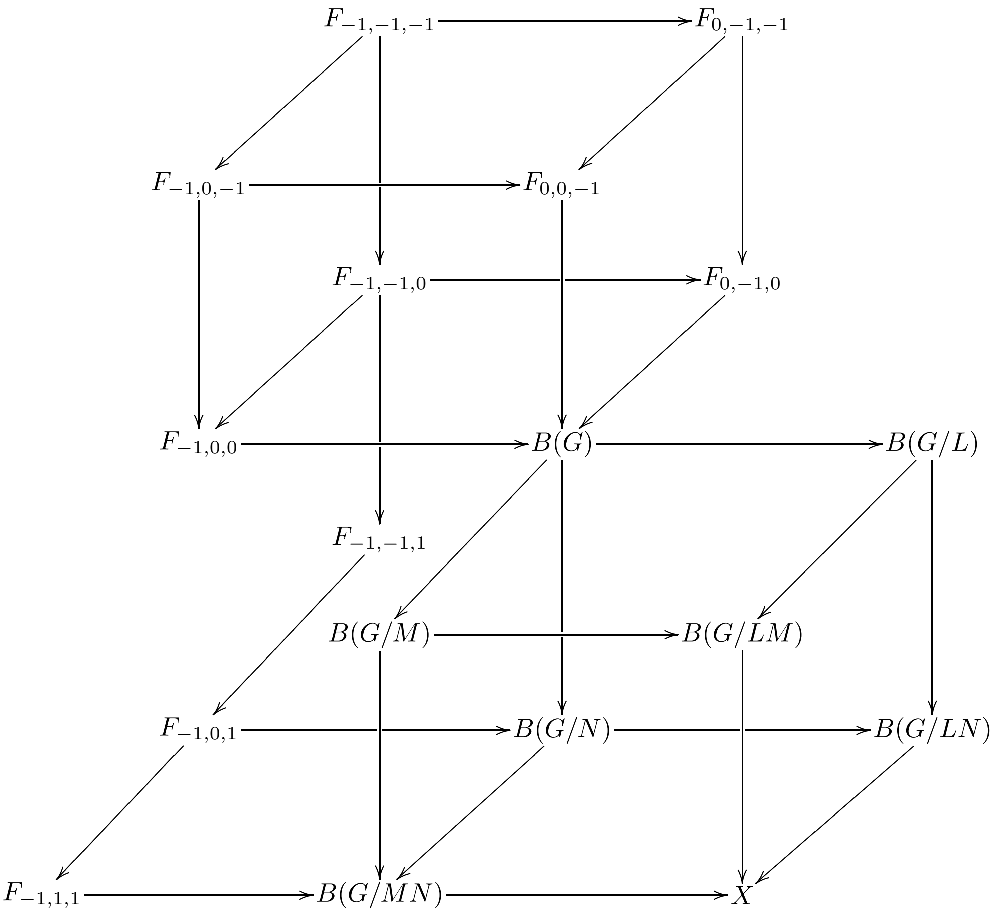

Figure 1

in which each row and each column is a fibration sequence. For precise details of the construction see [22, 8].

The homotopy exact sequence of the fibration $F_{-1,1,1} \to B(G/MN) \to X$ gives isomorphisms

$$\pi_n(X) \cong \pi_{n-1}(F_{-1,1,1}) \quad (n \geq 3). \tag{5}$$

The exact sequence of the fibration $F_{-1,-1,1} \to F_{-1,0,1} \to F_{-1,1,1}$ gives isomorphisms

$$\pi_2(F_{-1,1,1}) \cong \ker(\pi_1(F_{-1,-1,1}) \to L/(L \cap N)), \tag{6}$$

$$\pi_n(F_{-1,1,1}) \cong \pi_{n-1}(F_{-1,-1,1}) \quad (n \geq 4). \tag{7}$$

Note that

$$\pi_1(F_{-1,0,0}) \cong L \cap M. \tag{8}$$

Notation. For $x, y \in G$ we define $[x,y] = xyx^{-1}y^{-1}$ and ${}^x y = xyx^{-1}$. We write $A \sqcup B = < n >$ to mean that A, B are nonempty subsets of $\{1, \ldots, n\}$ with union equal to $\{1, \ldots, n\}$ and with trivial intersection. We write $A \sqcup B \sqcup C = < n >$ to mean that A, B, C are nonempty sets with union equal to $\{1, \ldots, n\}$ and with trivial pairwise intersections. We set $N_1 = L, N_2 = M, N_3 = N$ and $N_A = \cap_{i \in A} N_i$.

Brown and Loday's higher van Kampen theorem [8] asserts that a functor $\Pi\colon$ (3-cubes of spaces) $\to$ (cat^3-groups) preserves certain connectivity and colimits. In particular it asserts that all spaces in Figure 1 are path-connected. The following consequence of the colimit property was proved in [13] and published in the form of a more general result in [17].

Theorem 2 *[13, 17] The group $\pi_1(F_{-1,-1,-1})$ is isomorphic to the group $T(L,M,N)$ generated by symbols*

$$a \otimes_{A,B} b$$

for all $A \sqcup B =< 3 >, a \in N_A, b \in N_B$, subject to the relations

$$a \otimes_{A,B} b = (b \otimes_{B,A} a)^{-1},$$

$$aa' \otimes_{A,B} b = ({}^a a' \otimes_{A,B} {}^a b)(a \otimes_{A,B} b),$$

$$({}^u[u^{-1},v] \otimes_{U\cup V,W} {}^u w)({}^w[w^{-1},u] \otimes_{W\cup U,V} {}^w v)({}^v[v^{-1},w] \otimes_{V\cup W,U} {}^v u),$$

$$(a \otimes_{A,B} b)(a' \otimes_{A',B'} b')(a \otimes_{A,B} b)^{-1} = {}^{[a,b]}a' \otimes_{A',B'} {}^{[a,b]}b'$$

for $A \sqcup B = A' \sqcup B' =< 3 >, a \in N_A, a' \in N_{A'}, b \in N_B, b' \in N_{B'}, U \sqcup V \sqcup W =< 3 >, u \in N_U, v \in N_V, w \in N_W$. The homomorphism $T(L,M,N) \to \pi_1(F_{-1,-1,0}) = L \cap M$ maps $x \otimes y$ to the commutator $[x,y]$; this homomorphism has the structure of a crossed module $\partial\colon T(L,M,N) \to G, x \otimes y \mapsto [x,y]$ with action of $g \in G$ given by ${}^g(x \otimes y) = ({}^g x \otimes {}^g y)$.

The homotopy exact sequence of the fibration $F_{-1,-1,-1} \to F_{-1,-1,0} \to F_{-1,-1,1}$ together with isomorphisms (5)-(8) and Theorem 2 yield the following version of our main result for $n = 3$.

Theorem 3 *There are isomorphisms*

$$\pi_3(X) \cong \frac{L \cap M \cap N}{[L, M \cap N][M, L \cap N][N, L \cap M]},$$

$$\pi_4(X) \cong \ker(\partial\colon T(L,M,N) \to G).$$

For completeness we also include isomorphisms for the first two homotopy groups of X; these can both be deduced from Theorem 2 using the path-connectivity of the spaces in Figure 1 and the exact sequence of a fibration.

$$\pi_1(X) \cong \frac{G}{LMN}, \tag{9}$$

$$\pi_2(X) \cong \frac{LM \cap MN}{M(L \cap N)}. \tag{10}$$

Note that the left-hand side of (10) is symmetric in L, M, N. Hence the right-hand side is also symmetric in these three arbitrary normal subgroups of G.

The first isomorphism of the following corollary of Theorem 3 is a result of Wu [25].

Corollary 4 *Let $G = H * H$ be the free product of two copies of a group H. Let $\lambda, \mu, \nu\colon H * H \to H$ be the first projection, second projection and multiplication homomorphisms respectively. Set $L = \ker \lambda$, $M = \ker \mu$, $N = \ker \nu$. Then the suspension $SK(H,1)$ of a classifying space for H has*

$$\pi_3(SK(H,1)) = \frac{L \cap M \cap N}{[L, M \cap N][M, L \cap N][N, L \cap M]},$$

$$\pi_4(SK(H,1)) \cong \ker(\partial\colon T(L,M,N) \to G).$$

Proof. Let B be a classifying space for H. Then the wedge $B \vee B$ is a classifying space of $G = H * H$. Let CB denote the cone on B. Then $CB \vee B$ is a classifying space for G/L, $B \vee CB$ is a classifying space for G/M, and $B \times [0,1]$ is a classifying space for G/M. Moreover, the pushout (or union) of these classifying spaces is a suspension $X = SK(H,1)$. The corollary follows from Theorem 3. □

The first homomorphism of the following corollary was proved by Baues and Mikhailov [5].

Corollary 5 *Let K_1, K_2, K_3 be CW-subspaces of a connected CW-space K for which $K = K_1 \cup K_2 \cup K_3$ and $K^1 = K_1 \cap K_2 \cap K_3$ is the 1-skeleton of K. Let $F = \pi_1 K^1$ and $R_i = \ker(\pi_1 K^1 \to \pi_1 K_i)$ $(i = 1, 2, 3)$. Then there are homomorphisms*

$$\alpha_3 \colon \pi_3(K) \longrightarrow \frac{R_1 \cap R_2 \cap R_3}{[R_1, R_2 \cap R_3][R_2, R_1 \cap R_3][R_3, R_1 \cap R_2]},$$

$$\alpha_4 \colon \pi_4(K) \longrightarrow \ker(\partial \colon T(R_1, R_2, R_3) \to F).$$

In some cases both α and β are isomorphisms.

Proof. Each homomorphism $\pi_1 K^1 \to \pi_1 K_i$ is surjective since the 1-skeleton of K_i equals the 1-skeleton K^1 of K. Hence $\pi_1 K_i \cong F/R_i$. We can identify K^1 with the classifying space $B(F)$. There are maps $\iota_i \colon K_i \to B(F/R_i)$ inducing isomorphisms of fundamental groups. The maps ι_i induce a map $\iota \colon K = K_1 \cup K_2 \cup K_3 \to X$ where X is the colimit of Theorem 2 with $G = F, L = R_1, M = R_2, N = R_3$. The required homomorphisms α_k are obtained by composing the isomorphisms of Theorem 2 with the induced homomorphisms $\pi_k(\iota) \colon \pi_k(K) \to \pi_k(X)$ $(k = 3, 4)$.

Consider the 2-sphere $K = SK(\mathbb{Z}, 1)$ expressed as a union of $K_1 = CS^1 \vee S^1$, $K_2 = S^1 \vee CS^1$, $K_3 = S * 1 \times [0,1]$ as in the proof of Corollary 4. In this case the maps α_3, α_4 are isomorphisms. □

One can obtain group-theoretic results using Theorem 3. Let $\mathcal{X}$ be an alphabet and $F = F(X)$ be the free group generated by $\mathcal{X}$. Let $r_1, \ldots, r_n$ be words in F. The group presentation $\mathcal{P}(X,\ r_1, \ldots, r_n) = \langle X \mid r_1, \ldots, r_n \rangle$ is said to be *aspherical* if the associated 2-complex has trivial second homotopy group. We call the presentation *almost aspherical* if for every proper subset $\mathcal{S}$ of the set $\{r_1, \ldots, r_n\}$ the presentation $\langle X | \mathcal{S} \rangle$ is aspherical. For example, the 4-string braid group has the almost aspherical presentation $\langle x, y, z | xyx = yxy, yzy = zyz, xz = zx \rangle$.

Corollary 6 *Let $\mathcal{P}(X, r_1, r_2, r_3)$ be an almost aspherical presentation. Let R_i be the normal closure in $F = F(X)$ of the relator $\mathcal{R}_i$. Then the quotient group*

$$\frac{F}{[R_1, R_2 \cap R_3][R_2, R_1 \cap R_3][R_3, R_1 \cap R_2]}$$

is torsion free.

Proof. The 2-complexes associated to the presentations $\langle X \mid r_i \rangle$ and $\langle X \mid r_i,\ r_j \rangle$ are classifying spaces for F/R_i and $F/R_i R_j$ $(1 \le i \ne j \le 3)$. So on taking $G = F, L = R_1, M = R_2, N = R_3$ in Theorem 3 we can identify the colimit space X with the 2-complex associated to the three-relator presentation. Theorem 3 provides an exact sequence

$$0 \to \pi_3(X) \to \frac{F}{[R_1, R_2 \cap R_3][R_2, R_1 \cap R_3][R_3, R_1 \cap R_2]} \to \frac{F}{R_1 \cap R_2 \cap R_3} \to 1.$$

The universal cover of X is homotopy equivalent to a wedge of 2-spheres and thus $\pi_3(X)$ is known to be torsion free. The group $F/R_1 \cap R_2 \cap R_3$ maps into the group $(F/R_1) \times$

$(F/R_2) \times (F/R_3)$ and each F/R_i is torsion free since it has an aspherical presentation. The corollary follows. □

It is easily shown that $T(G,G,G)$ is isomorphic to the nonabelian symmetric square $G\tilde{\otimes}G$ introduced by Dennis [11] and then Brown and Loday [8]. In light of this isomorophism, the following corollary of Theorem 3 can be found in [8].

Corollary 7 *For the double suspension $S^2K(G,1)$ of a classifying space $K(G,1)$ there are isomorphisms*

$$\pi_3(S^2K(G,1)) \cong \frac{G}{[G,G]},$$

$$\pi_4(S^2K(G,1)) \cong \ker(\partial\colon T(G,G,G) \to G).$$

Proof. It suffices to observe that, in the case $L = M = N = G$, the colimit X is of the homotopy type of $S^2K(G,1)$. □

The above results on homotopy groups rely heavily on the homotopy exact sequence of a fibration. For calculations in group cohomology one defines $B(G,N)$ to be the homotopy cofibre of the map $B(G) \to B(G/N)$ and define $H_k(G) = H_k(B(G),\mathbb{Z})$, $H_k(G,N) = H_{k+1}(B(G,N),\mathbb{Z})$. This yields a long exact sequence

$$\cdots \to H_{k+1}(G/N) \to H_{k+1}(G,N) \to H_k(G) \to H_k(G/N) \to \cdots .$$

One defines $B(G,M,N)$ to be the homotopy cofibre of the canonical map $B(G,N) \to B(G/M, NM/M)$ and $H_k(G,M,N) = H_{k+2}(B(G,M,N),\mathbb{Z})$. There is thus a long exact sequence

$$\cdots \to H_{k+1}(G/M, MN/M) \to H_{k+1}(G,M,N) \to H_k(G,N) \to H_k(G/M, MN/N) \to \cdots .$$

The isomorphism in the following corollary was originally proved by a purely algebraic argument using the theory of nonabelian left derived functors [15]; however, the first proof to appear in print was a short topological one [7]. We recall the topological proof here.

Corollary 8 *Let R and S be normal subgroups of a free group F and suppose that both quotients F/R and F/S are free. Set $G = F/RS$. Then*

$$H_3(G) = \frac{R \cap S \cap [F,F]}{[R,S][R\cap S, F]}.$$

Proof. It is readily seen that, for any normal subgroups M, N in G, the cofibre $B(G,M,N)$ is homotopy equivalent to the above colimit X in the case $L = G$. Theorem 3 shows that

$$\pi_3(B(G,M,N)) \cong \frac{M \cap N}{[G, M\cap N][M,N]}.$$

Since $\pi_k(B(G,M,N)) = 0$ for $k = 1,2$, the Hurewicz isomorphism $\pi_3(B(G,M,N)) \cong H_3(B(G,M,N),\mathbb{Z})$ gives

$$H_1(G,M,N) \cong \frac{M \cap N}{[G, M\cap N][M,N]}.$$

Using $H_k(F) = H_k(F/R) = H_k(F/S) = 0$ for $k \geq 2$, the exact homology sequences of a cofibration yield an isomorphism

$$H_3(G) \cong \ker(H_1(F,R,S) \to H_1(F)).$$

This yields the corollary. □

The above formula for $H_1(G,M,N)$ is of independent interest. For a group-theoretic description of $H_2(G,M,N)$ we define the group $E(L,M,N)$ to be the quotient of $T(L,M,N)$ obtained by imposing the extra relations

$$x \otimes x = 1$$

for all $x \in L \cap M \cap N$. There is an induced crossed module $\partial\colon E(L,M,N) \to G$. The following result was proved algebraically in [15]. To keep things topological we outline a topological proof of the second isomorphism.

Corollary 9 *For a group G with normal subgroups M, N there are isomorphisms*

$$H_1(G,M,N) \cong \frac{M \cap N}{[G, M \cap N][M,N]},$$

$$H_2(G,M,N) \cong \ker(E(G,M,N) \to G).$$

Proof. Let $L = G$. We have already derived the first isomorphism from Theorem 3. For the second we consider J.H.C. Whitehead's exact sequence

$$H_5(X) \to \Gamma_4(X) \xrightarrow{\eta} \pi_4(X) \to H_4(X) \to 0.$$

A description of $\Gamma_4(X)$ is given in [4]; in our case we find

$$\Gamma_4(X) = ((M \cap N)/[G, M \cap N][M,N])) \otimes \mathbb{Z}_2.$$

To complete the proof one has to check that η is induced by the homomorphism $M \cap N \to T(G,M,N), x \mapsto x \otimes x$. □

The formulae in Theorem 3 and its corollaries can in some cases be implemented in a computer algebra system such as GAP [19] and used to obtain specific calculations. Suppose for example that G is a finite group. The integral homology group $H_n(G)$ is then finite for $n \geq 1$. The exact homology sequences relating $H_n(G)$, $H_n(G,N)$ and $H_n(G,M,N)$ imply that $H_n(G,M,N)$ is also finite. It thus follows from the proof of Corollary 9 that the group $T(G,M,N)$ is a finitely presented finite group and so one can attempt to use standard algorithms to compute it. The following special case formulae

$$\pi_4(S^2K(G,1)) = \ker(T(G,G,G) \to G),$$

$$H_2(G) = \ker(E(G,G,G) \to G),$$

$$\pi_3(SK(G,1)) = \ker(T(G,G) \to G),$$

(where $T(G,G)$ is defined in Theorem 12) have been implemented for finite groups G in the GAP package HAP [16]. As an illustration, the implementation can be used to compute

$$\pi_3(SK(GL_4(\mathbb{Z}_3))) = \pi_4(S^2K(GL_4(\mathbb{Z}_3))) = \mathbb{Z}_2, \quad H_2(GL_4(\mathbb{Z}_3)) = 0$$

for the group $GL_4(\mathbb{Z}_3)$ of 4×4 matrices over $\mathbb{Z}_3$; this group has order 24261120.

3 The case $n \geq 2$

We need to consider the category $\{0 < 1\}$ with two objects and one non-identity arrow $0 \to 1$. Following [22] we define an *n-cube of spaces* to be a functor $S\colon \{0 < 1\}^n \to$ (Spaces). We also need to consider the category $\{-1 < 0 < 1\}$ with three objects and two non-identity arrows $-1 \to 0$ and $0 \to 1$. An object in this category is an n-tuple $\Delta \in \{0,1\}^n$; we define a *row* to consist of a sequence of arrows $\Delta' \to \Delta \to \Delta$" where the n-tuples Δ', Δ, Δ" are identical except in say the ith coordinate. Following [22] we define an *n-cube of fibrations* to be a functor $F\colon \{-1 < 0 < 1\}^n \to$ (Spaces) that maps each row to a fibration sequence. It is explained in [22, 8] how an n-cube of spaces S gives rise to an n-cube of fibrations $F = \overline{S}$ where, up to homotopy type of maps, the restriction of $\overline{S}$ to $\{0 < 1\}^n$ is S.

Definition 10 *Following [8] we say that an n-cube of spaces S is connected if each space in the n-cube of fibrations $\overline{S}$ is path-connected.*

Given an n-tuple $\Delta \in \{0,1\}^n$ we shall write $i \in \Delta$ to mean that the ith coordinate of Δ eqials 1. To n normal subgroups $N_1, \ldots, N_n$ of a group G we associate the n-cube of spaces $B = B(G, N_1, \ldots, N_n)\colon \{0 < 1\}^n \to$ (Spaces) in which

$$B_\Delta = B(\frac{G}{\prod_{i\in\Delta} N_i})$$

is the classifying space of the quotient group $G/\prod_{i\in\Delta} N_i$, and the maps are induced by the canonical inclusions.

The verification of the following lemma, which uses the homotopy exact sequence of a fibration, is left to the reader.

Lemma 11 *Let $B = B(G, N_1, \ldots, N_n)$ be the n-cube of spaces associated to n normal subgroups of G.*

(i) The n-cube B is connected if $n = 1, 2$.

(ii) For $n \geq 3$ the n-cube B is connected if and only if for all subsets $I, J \subset \{1, \cdots, n\}$ with $|I| \geq 2, |J| \geq 1$ the following equality holds:

$$\left(\bigcap_{i\in I} N_i\right)\left(\prod_{j\in J} N_j\right) = \bigcap_{i\in I}\left(N_i(\prod_{j\in J} N_j)\right).$$

Let $B = B(G, N_1, \ldots, N_n)$. Let $\Delta_{\text{final}} = (1, 1, \ldots, 1)$ denote the n-tuple with each coordinate equal to 1. Let $\Delta_{\text{initial}} = (-1, -1, \ldots, -1)$ denote the n-tuple with each coordinate equal to -1. We are interested in the space X arising as the colimit of classifying spaces $\{B_\Delta : \Delta \in \{0 < 1\}^n, \Delta \neq \Delta_{\text{final}}\}$. Define $C = C(G, N_1, \ldots, N_n)\colon \{0 < 1\}^n \to$ (Spaces) to be the n-cube of spaces obtained from B by replacing the classifying space $B_{\Delta_{\text{final}}}$ with the colimit X. Let $F = \overline{C}$ be the n-cube of fibrations associated to C. The following theorem is a special case of Theorem 2.3 in [17] and is a consequence of Brown and Loday's higher van Kampen theorem.

Theorem 12 *[17] Let $N_1, \ldots, N_n$ be normal subgroups of G such that the (n-1)-cube of spaces $B(G, N_1, \ldots, \hat{N}_i, \ldots, N_n)$ is connected for each $1 \leq i \leq n$. Let $F = \overline{C}$ be the n-cube of fibrations associated to $C = C(G, N_1, \ldots, N_n)$. Every space in F is path-connected and $\pi_1(F_{\Delta_{\text{initial}}})$ is isomorphic to the group $T(N_1, \ldots, N_n)$ generated by symbols*

$$a \otimes_{A,B} b$$

for all $A \sqcup B = < n >, a \in N_A, b \in N_B$ *, subject to the relations*

$$a \otimes_{A,B} b = (b \otimes_{B,A} a)^{-1},$$

$$aa' \otimes_{A,B} b = ({}^a a' \otimes_{A,B} {}^a b)(a \otimes_{A,B} b),$$

$$({}^u[u^{-1}, v] \otimes_{U \cup V, W} {}^u w)({}^w[w^{-1}, u] \otimes_{W \cup U, V} {}^w v)({}^v[v^{-1}, w] \otimes_{V \cup W, U} {}^v u),$$

$$(a \otimes_{A,B} b)(a' \otimes_{A',B'} b')(a \otimes_{A,B} b)^{-1} = {}^{[a,b]}a' \otimes_{A',B'} {}^{[a,b]}b'$$

for $A \sqcup B = A' \sqcup B' = < n >, a \in N_A, a' \in N_{A'}, b \in N_B, b' \in N_{B'}, U \sqcup V \sqcup W = < n >$ *,* $u \in N_U, v \in N_V, w \in N_W$*. The homomorphism* $T(N_1, \dots, N_n) \to \pi_1(F_{-1,-1,\dots,-1,0}) \leq G$ *maps* $x \otimes y$ *to the commutator* $[x, y]$*; this homomorphism has the structure of a crossed module* $\partial \colon T(N_1, \dots, N_n) \to G, x \otimes y \mapsto [x, y]$ *with action of* $g \in G$ *given by* ${}^g(x \otimes y) = ({}^g x \otimes {}^g y)$*.*

Proof of Theorem 1. Our derivation of Theorem 3 from Theorem 2 extends routinely to yield our main Theorem 1 as a corollary of Theorem 12. One simply has to apply the homotopy exact sequence of a fibration n times. Lemma 11 provides the algebraic version of the connectivity condition. □

As already mentioned, Theorem 1 for the case $n = 2$ is given in [8]. This case yields the following earlier result of Brown [6]

Corollary 13 *[6] Given two normal subgroups* M, N *in a group* G *we have*

$$\pi_2(B(G/M) \cup_{B(G)} B(G/N)) = \frac{M \cap N}{[M, N]}.$$

Corollary 13 implies that $R_i \cap R_j = [R_i, R_j]$ $(i \neq j)$ in Corollary 6. Corollary 13 also imples that $L \cap M = [L, M], M \cap N = [M, N], L \cap N = [L, N]$ in Corollary 4. Furthermore, Corollary 13 can be used to recover the exact sequence of second homotopy groups proved by M. Gutierrez and J. Ratcliffe in [20].

With one exception, the corollaries in Section 3 and their proofs extend to analogous results involving $n \geq 2$ normal subgroups. The exception is Corollary 8 where we we do not know an explicit description of Whitehead's sequence in higher dimensions necessary for extending the topological proof.

4 Group-theoretical applications

Some unexpected results in group theory arise from well-known results in homotopy theory. For example Corollary 6 generalizes to the following.

Corollary 14 *Let* $\mathcal{P}(X, r_1, \dots, r_n)$ *be an almost aspherical presentation. Let* R_i *be the normal closure in* $F = F(X)$ *of the relator* R_i*. Then the quotient group*

$$\frac{F}{\prod_{I \cup J = \{1,\dots,n\},\ I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j]}$$

can only have p*-torsion if the* n*th homotopy group of a wedge of spheres can have* p*-torsion. So for example, by a result of Serre, the group can have no* p *torsion for primes* $p > 2n$*.*

Proof Consider a proper subset $I \subsetneq \{1, \ldots, n\}$. Since the group presentation $\mathcal{P}(X, r_i \ (i \in I))$ is aspherical the associated 2-complex K_I is a classifying space for the group $F/\prod_{i\in I} R_i$. Also, this space K_I is the homotopy colimit (i.e. union) of the diagram of spaces $\{B(F/\prod_{i\in J} R_j) : J \subsetneq I\} = \{K_J : J \subsetneq I\}$. Using induction on the size of I together with the connectivity assertion of Theorem 12 we obtain that the $|I|$-cube of spaces $B(F, R_i \ (i \in I))$ is connected. In particular each $(n-1)$-cube $B(F, R_1, \ldots, \hat{R}_i, \ldots, R_n)$ is connected and so we can apply Theorem 1 to the normal subgroups $(F, R_1, \ldots, R_n)$; we get a formula for the n-th homotopy group of a colimit space which, in this case, is the 2-complex K associated to the presentation $\mathcal{P}(X, r_1, \ldots, r_n)$. Now $\pi_2 K \cong \pi_2 \tilde{K}$ and $\tilde{K}$ is a wedge of 2-spheres. The remainder of the proof is now just a copy of the proof of Corollary 6. □

For a second example we recall a description of the homotopy groups of the 2-sphere due to Wu [25]. Let $F[S^1]$ be Milnor's $F[K]$-construction applied to the simplicial circle S^1. This is the free simplicial group with $F[S^1]_n$ a free group of rank $n \geq 1$ with generators $x_0, \ldots, x_{n-1}$. Changing the basis of $F[S^1]_n$ in the following way: $y_i = x_i x_{i+1}^{-1}$, $y_{n-1} = x_{n-1}$, we get another basis $\{y_0, \ldots, y_{n-1}\}$ in which the simplicial maps can be written more easily. A combinatorial group-theoretical argument then gives a description of the n-th homotopy group of the loop space $\Omega\Sigma S^1$, which is isomorphic to the homotopy group of $\pi_{n+1}(S^2)$ (see [25] for precise details); one finds Wu's formula

$$\pi_{n+1}(S^2) \cong \frac{\langle y_{-1}\rangle^F \cap \langle y_0\rangle^F \cap \cdots \cap \langle y_{n-1}\rangle^F}{[[y_{-1}, y_0, \ldots, y_{n-1}]]}, \ n \geq 1 \tag{11}$$

where F is a free group with generators $y_0, \ldots, y_{n-1}$, $y_{-1} = (y_0 \ldots y_{n-1})^{-1}$, and $[[y_{-1}, y_0, \ldots, y_{n-1}]]$ is the normal closure in F of the set of left-ordered commutators

$$[z_1^{\varepsilon_1}, \ldots, z_t^{\varepsilon_t}] \tag{12}$$

with the properties that $\varepsilon_i = \pm 1$, $z_i \in \{y_{-1}, \ldots, y_{n-1}\}$ and all elements in $\{y_{-1}, \ldots, y_{n-1}\}$ appear at least once in the sequence of elements z_i in (12).

Consider $n \geq 1$ and Milnor's construction $F[S^n]$. The lower terms of the simplicial group $F[S^n]$ are the following:

$$\begin{aligned}
&F[S^n]_n = F(\sigma),\\
&F[S^n]_{n+1} = F(s_0\sigma, \ldots, s_n\sigma),\\
&F[S^n]_{n+2} = F(s_j s_i \sigma \mid i < j),\\
&\ldots
\end{aligned}$$

For $n \geq 1$, the generator of

$$\pi_{n+1}(F[S^n]) \simeq \pi_{n+1}(\Omega(S^{n+1})) \simeq \pi_{n+2}(S^{n+1})$$

can be chosen as the commutator $[s_0\sigma, s_1\sigma]$ in $F[S^n]_{n+1}$ (see [25]). Now these commutators considered as elements of $F[S^n]_{n+1}$ define the elements from $F[S^1]_k$, which correspond the homotopy classes of composition maps

$$S^{k+1} \to S^k \to \cdots \to S^3 \to S^2,$$

where every map is viewed as a suspension over the Hopf fibration. Let us consider these elements.

1. First, let $F_2 = F(y_0, y_1)$, then the element

$$[y_0, y_1] \notin [[y_{-1}, y_0, y_1]]$$

corresponds to the homotopy class of the Hopf fibration $S^3 \to S^2$.

2. Let $F_3 = F(y_0, y_1, y_2)$, then the element

$$[[y_0, y_1], [y_0, y_1y_2]] \notin [[y_{-1}, y_0, y_1, y_2]]$$

corresponds to the homotopy class of the composition map $S^4 \to S^3 \to S^2$.

3. Let $F_4 = F(y_0, y_1, y_2, y_3)$, then the element

$$[[[y_0, y_1], [y_0, y_1y_2]], [[y_0, y_1], [y_0, y_1y_2y_3]]] \notin [[y_{-1}, y_0, y_1, y_2, y_3]]$$

corresponds to the homotopy class of the composition map $S^5 \to S^4 \to S^3 \to S^2$.

4. Let $F_5 = F(y_0, y_1, y_2, y_3, y_4)$, then the element

$$\begin{aligned}&[[[[y_0, y_1], [y_0, y_1y_2]], [[y_0, y_1], [y_0, y_1y_2y_3]]],\\ &\qquad [[[y_0, y_1], [y_0, y_1y_2]], [[y_0, y_1], [y_0, y_1y_2y_3y_4]]]]\\ &\qquad\qquad \notin [[y_{-1}, y_0, y_1, y_2, y_3, y_4]]\end{aligned}$$

corresponds to the homotopy class of the composition map $S^6 \to S^5 \to S^4 \to S^3 \to S^2$.

5. Let $F_6 = F(y_0, y_1, y_2, y_3, y_4, y_5)$, then the element

$$\begin{aligned}&[[[[[y_0, y_1], [y_0, y_1y_2]], [[y_0, y_1], [y_0, y_1y_2y_3]]],\\ &\qquad [[[y_0, y_1], [y_0, y_1y_2]], [[y_0, y_1], [y_0, y_1y_2y_3y_4]]]],\\ &\qquad [[[[y_0, y_1], [y_0, y_1y_2]], [[y_0, y_1], [y_0, y_1y_2y_3]]],\\ &\qquad [[[y_0, y_1], [y_0, y_1y_2]], [[y_0, y_1], [y_0, y_1y_2y_3y_4y_5]]]]]\\ &\qquad\qquad \in [[y_{-1}, y_0, y_1, y_2, y_3, y_4, y_5]]\end{aligned}$$

corresponds to the trivial homotopy class of the composition map

$$S^7 \to S^6 \to S^5 \to S^4 \to S^3 \to S^2. \tag{13}$$

The triviality of this map can be proved using standard methods in homotopy theory [24]. This is the simplest case of the Nilpotence Theorem due to Nishida [23], which states that every element in the ring of stable homotopy groups of spheres is nilpotent.

We now turn to a group-theoretic application of Theorem 1. Consider a free group $F(X)$ with a basis X and let $\mathcal{S} = \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$ be a collection of sets $\mathcal{R}_i$ of words, such that: 1) the presentation

$$\langle X \mid \mathcal{R}_1, \dots, \mathcal{R}_n \rangle \tag{14}$$

is not aspherical, 2) the presentation

$$\langle X \mid \mathcal{R}_{i_1}, \dots, \mathcal{R}_{i_k} \rangle \tag{15}$$

is aspherical for every proper subset $\{\mathcal{R}_{i_1}, \dots, \mathcal{R}_{i_k}\}$ of $\mathcal{S}$. Such a presentation we shall call *$\mathcal{S}$-almost aspherical*. Observe that every almost aspherical presentation is $\mathcal{S}$-almost aspherical, where $\mathcal{S}$ is the collection of all relators, i.e. the sets $\mathcal{R}_i$ contain single relators for all i.

Denote by R_i the normal closure of $\mathcal{R}_i$ for $i = 1, \dots, n$. Observe that the standard 2-complex, associated to an $\mathcal{S}$-almost aspherical presentation (14) is homotopically equivalent to the homotopy colimit X of the n-cube of standard 2-complexes associated with subpresentations of the type (15), i.e. to the colimit of the non-final spaces in the n-cube $B(F, R_1, \dots, R_n)$. The asphericity condition 2) implies that the above n-cube is n-connected and hence the n-th homotopy group $\pi_n(X)$ can be computed by formula given in Theorem 1. However, the colimit X is homotopically equivalent to a (non-empty) wedge of 2-spheres and, therefore, formulas for the homotopy groups of wedges of 2-spheres follow.

As an example, consider an almost aspherical presentation of the trivial group:

$$\mathcal{P} = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n+1} \rangle$$

with n generators and $n+1$ relators ($n \geq 1$). Let $K_{\mathcal{P}}$ be the standard 2-complex associated with $\mathcal{P}$. The chain complex $C_* K_{\mathcal{P}}$ has $C_2 K_{\mathcal{P}} = \mathbb{Z}^{n+1}$, $C_1 K_{\mathcal{P}} = \mathbb{Z}^n$ and natural epimorphism $C_2 K_{\mathcal{P}} \to C_1 K_{\mathcal{P}}$, which kernel is isomorphic to $H_2(K_{\mathcal{P}}) = \pi_2(K_{\mathcal{P}}) = \mathbb{Z}$, hence $K_{\mathcal{P}}$ is homotopically equivalent to the 2-sphere S^2. Theorem 1 implies the following presentation of the $(n+1)$-st homotopy group of S^2:

$$\pi_{n+1}(S^2) \simeq \frac{R_1 \cap \dots \cap R_{n+1}}{\prod_{I \cup J = \{1, \dots, n\},\ I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j]}, \tag{16}$$

where $R_i = \langle r_i \rangle^F$, F being the free group with generators $x_1, \dots, x_n$. Clearly, in this case, the subgroup of $F/(\prod_{I \cup J = \{1, \dots, n\},\ I \cap J = \emptyset} [\cap_{i \in I} R_i, \cap_{j \in J} R_j])$ given in the right hand side of (16) is central. In particular, consider the presentation

$$\mathcal{P}_{WU} = \langle x_1, \dots, x_n \mid x_1, \dots, x_n, x_1 \cdots x_n \rangle.$$

It is easy to see that $\mathcal{P}_{WU}$ is almost aspherical. A standard commutator calculus argument, given essentially in Corollary 3.5 of [25] shows that

$$[[x_1 \dots x_n, x_1, \dots, x_n]] = \prod_{I \cup J = \{1, \dots, n+1\},\ I \cap J = \emptyset} [\bigcap_{i \in I} R_i, \bigcap_{j \in J} R_j], \tag{17}$$

and Wu's isomorphism (11) follows from (16). (One could also obtain (17) using Theorem 1.)

Now consider an arbitrary set of elements $r_1, \dots, r_{n+1}$ in the free group F with basis $x_1, \dots, x_n$ and with the following property: the groups defined by presentations with n generators and n relators

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_{j-1}, \hat{r}_j, r_{j+1}, \dots, r_{n+1} \rangle$$

are trivial for all $j = 1, \dots, n+1$ (i.e. we consider the presentation without symbol r_j). Since every balanced presentation, i.e. a presentation with equal number of generators and relators, of the trivial group is aspherical, we have the following: *either $\pi_{n+1}(S^2)$ can be presented as (16) for $R_i = \langle r_i \rangle^F,\ i = 0, \dots, n+1$, or there is a counter-example to Whitehead asphericity conjecture, i.e. there exists an aspherical 2-dimensional complex with non-aspherical subcomplex.*

Since every subcomplex of an aspherical complex, with a single 2-cell, is aspherical (see [1]), we obtain the following. Let F be a free group with generators x_1, x_2 and let r_1, r_2, r_3 be words in F such that the groups $F/R_1R_2, F/R_1R_3, F/R_2R_3$ are trivial for $R_i = \langle r_i \rangle^F,\ i = 1, 2, 3$. Then there is an isomorphism

$$\frac{R_1 \cap R_2 \cap R_3}{[R_1, R_2 \cap R_3][R_2, R_1 \cap R_3][R_3, R_1 \cap R_2]} \simeq \mathbb{Z}.$$

As an example, consider

$$r_1 = x_1^2 x_2^{-3},\ r_2 = x_1^3 x_2^{-4},\ r_3 = x_1 x_2 x_1 x_2^{-1} x_1^{-1} x_2^{-1}.$$

The groups

$$\langle x_1, x_2 \mid x_1^n x_2^{-(n+1)},\ x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}\rangle$$

are trivial for $n \geq 2$. The above presentation of trivial groups is due to Akbulut-Kirby [2]. It is easy to see that the presentation

$$\langle x_1, x_2 \mid x_1^2x_2^{-3},\ x_1^3x_2^{-4}\rangle$$

defines the trivial group. It would be interesting to find a natural generalization of this presentation to the case of n generators and $n+1$ relators and to consider group-theoretical meaning of the fact that the composition map (13) is homotopically trivial.

5 Other varieties

The Hopf type formula of Corollary 8 and its higher dimensional analogues were first proved using a modification of a technique of F. Keune for nonabelian derived functors. The technique is very general and can be applied in situations such as the homology of Lie algebras or varietal Baer invariants of groups. Indeed, it was used in [9] to derive a Hopf type formula for the Baer invariant corresponding to the variety of 2-nilpotent groups; this formula was implemented on a computer and applied to all groups of order at most 30.

The nonabelian derived functor approach and the generalised van Kampen theorem approach to Hopf type formulae both apply only to connected n-cubes (see Definition 10 and Lemma 11). Unfortunately this hypothesis was omitted from the statement of theorems in [15] and [7] and some subsequent publications. This seems to have caused confusion. However, in all calculations based on the theorems the connectivity condition was met. A detailed erratum [14] has been available on the first author's home page since 2002.

6 Acknowledgments

We are grateful to R. Brown for his comments on an earlier draft of this paper. We are also grateful to the Max Planck Institute für Mathematik, Bonn for inviting both authors to a symposium in honour of H.-J. Baues; work on the paper began during this symposium.

References

[1] J. Adams: "A new proof of a theorem of W. H. Cockcroft", J. London Math. Soc., 30 (1955), 482-488.

[2] S. Akbulut and R. C. Kirby, "A potential smooth counterexample in dimension four to the Poincare conjecture, the Schoenfliess conjecture and the Andrews-Curtis conjecture, Topology 24 (1985), 375-390.

[3] A. Bak, G. Donadze, N. Inassaridze, M. Ladra, "Homology of multiplicative Lie rings", J. Pure Applied Algebra 208 (2007) 761-777.

[4] H.-J. Baues, Homotopy type and homology, Oxford Mathematical Monographs (1996).

[5] H.-J. Baues and R. Mikhailov, "Intersections of subgroups in free groups and homotopy groups", to appear Int. J. Alg. Comp.

[6] R. Brown, "Coproducts of crossed P-modules", Topology 1984.

[7] R. Brown and G. Ellis, "Hopf formulae for the higher homology of a group", Bull. London Math. Soc. 20 (1988) 124-128.

[8] R. Brown and J.-L. Loday, "Van Kampen theorems for diagrams of spaces", Topology 26 (1987) 311-335.

[9] J. Burns and G. Ellis, "Nilpotent multipliers", Math. Zeitschrift 1993.

[10] J.M. Casas, N. Inassaridze, E. Khmaladze, M. Ladra, "Homology of $(n+1)$-types and Hopf type formulae", J. Pure Applied Algebra 200 (2005) 267-280.

[11] K. Dennis, "In search of new homology functors having a close relationship to K-theory", Cornell preprint 1976.

[12] G. Donadze, N. Inassaridze, T. Porter, "N-fold Cech derived functors and generalized Hopf type formulae", K-Theory 35 (2006) 341-373.

[13] G. Ellis, "Crossed modules and their higher dimensional analogues", PhD thesis, Bangor (1984).

[14] G. Ellis, "An erratum on Hopf type formulae", http://hamilton.nuigalway.ie

[15] G. Ellis, "Relative derived functors and the homology of groups", Cahier Topologie et Géometrie Diff. Cat. XXXI-2 (1990) 121-135.

[16] G. Ellis, "Homological Algebra Programming", AMS Contemprary Mathematics, to appear.

[17] G. Ellis and R. Steiner, "Higher-dimensional crossed modules and the homotopy groups of $(n+1)$-ads", J. Pure Applied Algebra 46 (1987) 117-136.

[18] T. Everaert, M. Gran and T. Van der Linden, "Higher Hopf formulae for homology via Galois Theory", Advances in Math. (to appear).

[19] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.9; 2006. (http://www.gap-system.org)

[20] M.A. Gutierrez and J. Ratcliffe, "On the second homotopy group", Quarterly J. Math. Oxford 32 (1981) 45-55.

[21] F. Keune, "The relativisation of K_2", J. Algebra 54 (1978) 159-177.

[22] J.-L. Loday, "Spaces with finitely many non-trivial homotopy groups", J. Pure Applied Algebra 24 (1982) 179-202.

[23] G. Nishida, "The nilpotency of elements of the stable homotopy groups of spheres", Journal of the Mathematical Society of Japan 25 (1973), 707-732.

[24] H. Toda: "Composition methods in homotopy groups of spheres", Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J. (1962).

[25] J. Wu, "Combinatorial description of homotopy groups of certain spaces", Math. Proc. Cambridge Phyl. Soc. 130 (2001) 489-