Homological Algebra Programming

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1. Introduction

This article is an overview of some attempts at using GAP [**11**] and Polymake [**12**] software for basic calculations in group cohomology. Section 2 contains examples, obtained using the HAP package [**13**] for the GAP system, which illustrate the potential of computational techniques in this area. Subsequent sections briefly outline the methods underlying these examples. Section 3 explains how an element of choice, present in many homological algebra constructions, can be made algorithmic using the notion of contracting homotopy. Section 4 describes a linear algebraic approach to computing the cohomology of small groups. Section 5 discusses five direct geometric methods for obtaining the integral cohomology of a range of finite and infinte groups. The final section considers two methods for computing the cohomology of a group G from that of certain subgroups of G.

We begin with some basic definitions. Let k be an integral domain and let $A = k[x_1, \ldots, x_n]/I$ be the quotient of a free associative ring by a two-sided ideal I. Let M be an A-module. A sequence of A-module homomorphisms

$$\cdots \xrightarrow{d_4} R_3 \xrightarrow{d_3} R_2 \xrightarrow{d_2} R_1 \xrightarrow{d_1} R_0$$

is said to be a *free A-resolution* of M if

- *(Exactness)* $\ker d_n = \mathrm{image}\, d_{n+1}$ for all $n \geq 1$,
- *(Freeness)* R_n is a free A-module for all $n \geq 0$,
- *(Augmentation)* the cokernel of d_1 is isomorphic to the module M.

Given such a resolution R_* and an A-module N one defines

$$\mathrm{Ext}^n_A(M,N) = \frac{\ker(\mathrm{Hom}_A(R_n,N) \to \mathrm{Hom}_A(R_{n+1},N))}{\mathrm{image}(\mathrm{Hom}_A(R_{n-1},N) \to \mathrm{Hom}_A(R_n,N))}$$

and

$$\mathrm{Tor}_n^A(M,N) = \frac{\ker(R_n \otimes_A N \to R_{n-1} \otimes_A N)}{\mathrm{image}(R_{n+1} \otimes_A N \to R_n \otimes_A N)}.$$

It can be shown that, up to isomorphism, the functors $\mathrm{Ext}^n_A(M,N)$ and $\mathrm{Tor}_n^A(M,N)$ do not depend on the choice of resolution R_*.

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Commutative algebra software such as CoCoA [**9**], Macaulay [**18**] and Singular [**22**] contains a range of Gröbner basis methods for computing these functors in the case where k is a field and the ring A is commutative. The Plural [**22**] extension to Singular handles certain non-commutative rings A.

To define the cohomology of a group G one takes the ring of integers $k = \mathbb{Z}$, the module $M = \mathbb{Z}$ with trivial G-action, the group ring $A = \mathbb{Z}G$, and sets

$$H^n(G,N) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},N), \quad H_n(G,N) = \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},N).$$

The GAP and Magma computational algebra systems contain methods for computing group cohomology in dimensions $n = 1$ and $n = 2$ for a range of groups G and modules N. Magma also has methods for the higher-dimensional cohomology of small p-groups with coefficients in the field of p elements, i.e., $N = \mathrm{GF}(p)$.

The computation of group cohomology involves two computationally expensive but independent tasks: the computation of a free resolution, and the computation of the homology of a chain complex. For the latter one can use GAP's internal Smith Normal Form function or more specialized Smith Normal Form algorithms avaiable in the GAP packages EDIM [**17**] and SIMPHOM [**10**]. Sections 3–6 below focus on the former task.

2. Example computations

The following automated proofs can be reproduced using the computer algebra system GAP with the HAP package loaded.

Theorem 2.1 ([**19**]). *The Mathieu group M_{23} has trivial integral homology $H_n(M_{23},\mathbb{Z}) = 0$ in dimensions $n = 1, 2, 3$.*

Proof.
```
gap> GroupHomology(MathieuGroup(23),1);
[  ]
gap> GroupHomology(MathieuGroup(23),2);
[  ]
gap> GroupHomology(MathieuGroup(23),3);
[  ]
```

Explanation of proof. The HAP package includes a function `GroupHomology(G,n)` which inputs a group G and integer $n \geq 1$. The function checks to see if G is finite, infinite, abelian, small order, nilpotent, crystallographic, Artin etc. On the basis of this crude data the function tries to decide on an appropriate method for constructing $(n + 1)$-terms of a free $\mathbb{Z}G$-resolution. It applies the method, then tensors the constructed resolution with the trivial module $\mathbb{Z}$, and finally chooses a Smith Normal Form algorithm to compute the n-th homology of the resulting chain complex. This homology is, by definition, the integral homology of G. The homology is returned as a list of its abelian invariants. In the above proof the lists are empty because the homology is trivial. □

Most groups G can be viewed in many diferent ways and the choice of the most appropriate methods for computing homology is a difficult one and often needs experimentation. The command `GroupHomology(G,n)` is a composite of several more basic HAP functions and attempts, in a fairly crude way, to make reasonable choices for parameters in the calculation of group homology. For any particular group G better results can usually be obtained by using the more basic functions directly with the user's choice of parameters.

Theorem 2.1 was originally proved in [**19**] as a counter-example to a long standing conjecture (attributed in [**19**] to J.-L. Loday) that if a finite group has trivial integral homology in the first three dimensions then the group must be trivial. The paper [**19**] also shows that $H_4(M_{23}, \mathbb{Z}) = 0$.

THEOREM 2.2. [**4, 20**] *(i) The group* $K_3 = \ker(\mathsf{SL}_2(\mathbb{Z}_{3^3}) \to \mathsf{SL}_2(\mathbb{Z}_3))$ *has third integral homology group of exponent* 27. *(ii) In dimensions* $n \neq 3$, $1 \leq n \leq 6$ *the group* K_3 *has integral homology of exponent at most* 9.

PROOF.

```
gap> K3:=MaximalSubgroups(
>          SylowSubgroup(SL(2,Integers mod 3^3),3))[2];;
gap> K3:=Image(IsomorphismPcGroup(K3));;
gap> Display(List([1..4],n->GroupHomology(K3,n)));
[ [   3,   3,   3 ],
  [   3,   3,   3 ],
  [   3,   3,   3,   3,   3,   3,  27 ],
  [   3,   3,   3,   3,   3,   3,   3,   3 ],
  [   3,   3,   3,   3,   3,   3,   3,   3,   3,   3,   9,
        9,   9 ],
  [   3,   3,   3,   3,   3,   3,   3,   3,   3,   3,   9,
        9,   9,   9,   9 ] ]
```

□

Theorem 2.2(i) was originally proved in [**4**] with the prime 3 replaced by any odd prime. The statement of Theorem 2.2(ii) was proved for all dimensions $n \neq 3$ in [**20**] using a technique of Ian Leary. The papers [**4, 20**] provided a counter-example to A. Adem's long standing conjecture that if the integral homology of a finite p-group has exponent e in some dimension then it has exponent e in infinitely many dimensions.

THEOREM 2.3. [**25**] *The mod* 2 *cohomology* $H^n(M_{11}, \mathbb{Z}_2)$ *of the Mathieu group* M_{11} *is a vector space of dimension equal to the coefficients of* x^n *in the Poincaré series* $(x^4 - x^3 + x^2 - x + 1)/(x^6 - x^5 + x^4 - 2 * x^3 + x^2 - x + 1)$ *for all* $n \leq 20$.

PROOF.

```
gap> PoincareSeriesPrimePart(MathieuGroup(11),2,20);
(x^4-x^3+x^2-x+1)/(x^6-x^5+x^4-2*x^3+x^2-x+1)
```

Explanation of proof. The HAP function `PoincareSeriesPrimePart(G,p,N)` inputs a finite group G, a prime p and a positive integer N. It returns a series $p(x)$ in which the coefficient of x^n equals the dimension of $H^n(G, \mathbb{Z}_p)$ for all $n \leq N$. This equality is not guaranteed for $n > N$. (However, since the cohomology ring $H^*(G, \mathbb{Z}_p)$ is finitely generated, there definitely exists some unkown value N_0 such that if the coefficients of the polynomial $p(x)$ are correct for all $n < N_0$ then the coefficients of $p(x)$ are correct for all n.) □

Theorem 2.3 was originally proved in [**25**] for all $n \geq 1$. We should emphasize that, currently, HAP can only prove Theorem 2.3 for values of n in some finite range such as $0 \leq n \leq 20$.

THEOREM 2.4. [**23**] *The symmetric group $G = S_3$ admits a periodic free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ of period 4.*

PROOF.
```
gap> F:=FreeGroup(2);; x:=F.1;; y:=F.2;;
gap> S3:=F/[ x^2, x*y*x^-1*y^-2 ];;
gap> R:=ResolutionSmallFpGroup(S3,5);;
gap> List([1..5],i->R!.dimension(i));
[ 2, 2, 1, 1, 2 ]
gap> R!.boundary(5,1)=R!.boundary(1,1);
true
gap> R!.boundary(5,2)=R!.boundary(1,2);
true
```
Explanation of proof. The HAP function `ResolutionSmallFpGroup(G,n)` inputs a finitely presented group G and a positive integer n. It returns a record containing the description of n terms of a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. The resolution corresponds to the cellular chain complex of the universal cover of a classifying CW-space $X = K(G, 1)$; the 2-skeleton of X is the usual 2-complex associated to the given presentation of G. The last two commands in the proof show that the fifth boundary map of the resolution equals the first. The resolution can thus be extended to a periodic one. □

Theorem 2.4 was first proved in [**23**]. Groups which act freely on spheres admit periodic resolutions. The interest in the group S_3 is that it does not act freely on a sphere.

THEOREM 2.5. [**6**] *The quaternion group Q of order 8 admits a classifying CW-space with one cell in dimension $4n$, three cells in dimension $4n + 1$, four cells in dimension $4n + 2$, and two cells in dimension $4n + 3$. The 2-skeleton of X corresponds to the presentation $Q = \langle i, j, k : ij = k, jk = i, ki = j, ikj = 1\rangle$.*

PROOF.
```
gap> A:=[[ 0,-1,0,0,],[1,0,0,0,],[0,0,0,1],[0,0,-1,0]];;
gap> B:=[[ 0,0,-1,0],[0,0,0,-1],[1,0,0,0],[0,1,0,0]];;
gap> Q:=Group(A,B);; P:=PolytopalComplex(Q,[1,0,0,0]);;
gap> ranks:=List([0..3],n->Dimension(P)(n));
[ 1, 3, 4, 2 ]
gap> List([1..3],n->List([1..ranks[n+1]],
>             k->Order(P!.stabilizer(n,k))));
[ [ 1, 1, 1 ], [ 1, 1, 1, 1 ], [ 1, 1 ] ]
gap> PresentationOfResolution(P);
rec( freeGroup := <free group on the generators [ f1, f2, f3 ]>,
  relators := [ f2*f3^-1*f1^-1, f3*f2*f1^-1,
                          f1*f2*f3, f1*f3^-1*f2 ] )
```
Explanation of proof. The group Q is entered as a group generated by two 4×4 matrices. The HAP function `PolytopalComplex(G,v)` inputs a finite group G of $n \times n$ rational matrices and a rational vector v of length n. The function returns a record describing the cellular chain complex of the convex hull of the orbit of v under the action of G. The group G acts on this convex hull, and the record contains the stabilizer groups of the various faces in this polytope. If the stabilizers are trivial

then the cellular chain complex forms part of a periodic free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. The HAP function `PresentationOfResolution(R)` returns the group presentation corresponding to the first two dimensions of a geometric resolution R. □

Theorem 2.5 yields a slightly different periodic free $\mathbb{Z}Q$-resolution of $\mathbb{Z}$ to that given by Cartan and Eilenberg [**6**]. In particular, its 2-skeleton corresponds to the 'natural' presentation of the quaternions.

THEOREM 2.6. *The symmetric group S_4 admits a classifying CW-space whose 2-skeleton corresponds to the Coxeter presentation*

$$S_4 = \langle x, y, z : x^2 = y^2 = z^2 = (xz)^2 = (xy)^3 = (yz)^3 = 1\rangle$$

and which has precisely 97 cells in dimension 20.

PROOF.

```
gap> R:=ResolutionFiniteGroup(SymmetricGroup(4), 20);;
gap> P:=PresentationOfResolution(R);
[ <free group on the generators [ f1, f2, f3 ]>,
  [ f1^2, f2^2, f3^2, f3*f1*f3*f1,
    f1*f2*f1*f2*f1*f2, f2*f3*f2*f3*f2*f3 ] ]
gap>  Dimension(R)(20);
97
```

Explanation of proof. The HAP function `ResolutionFiniteGroup(G,n)` constructs the first n terms of a free $\mathbb{Z}G$-resolution of a finite group G; the resolution is isomorphic to the cellular chain complex of the universal covering space of some classifying CW-space for G. □

Theorem 2.6 answers negatively a question in [**21**] which asks whether any classifying space for an n generator Coxeter group G, whose 2-skeleton corresponds to the standard Coxeter presentation of G, must have at least $\frac{(n+k-1)!}{(n-1)!k!}$ k-dimensional cells.

THEOREM 2.7. *(i)* [**15**] *The free nilpotent group G of class two on 4 generators has integral cohomology groups*

$$\begin{array}{lll} H^1(G,\mathbb{Z}) \cong \mathbb{Z}^4, & H^2(G,\mathbb{Z}) \cong \mathbb{Z}^{20}, & H^3(G,\mathbb{Z}) \cong \mathbb{Z}^{56}, \\ H^4(G,\mathbb{Z}) \cong \mathbb{Z}^{84}, & H^5(G,\mathbb{Z}) \cong \mathbb{Z}_3^4 \oplus \mathbb{Z}^{90}, & H^6(G,\mathbb{Z}) \cong \mathbb{Z}_3^4 \oplus \mathbb{Z}^{84}, \\ H^7(G,\mathbb{Z}) \cong \mathbb{Z}^{56}, & H^8(G,\mathbb{Z}) \cong \mathbb{Z}^{20}, & H^9(G,\mathbb{Z}) \cong \mathbb{Z}^4, \\ H^{10}(G,\mathbb{Z}) \cong \mathbb{Z}, & H^n(G,\mathbb{Z}) = 0 \ (n \geq 11). & \end{array}$$

(ii) The ring $H^*(G,\mathbb{Z})$ is generated by 4 classes in degree 1, 20 classes in degree 2, 36 classes in degree 3, and 20 classes in degree 4.

PROOF. To save space we show how to prove the completely analogous result for the free nilpotent group of class two on $n = 3$ generators. The only modification needed to get the above theorem is to set $n = 4$ and $m = 11$.

```
gap> n:=3;;m:=7;;
gap> F:=FreeGroup(3);;G:=NilpotentQuotient(F,2);;
gap> R:=ResolutionNilpotentGroup(G,10);;
gap> for n in [1..m-1] do
```

```
> Print(``Cohomology in dimension '',n,`` = '',
> Cohomology(HomToIntegers(R),n),``\n''); od;
Cohomology in dimension 1 = [ 0, 0, 0 ]
Cohomology in dimension 2 = [ 0, 0, 0, 0, 0, 0, 0, 0 ]
Cohomology in dimension 3 = [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
Cohomology in dimension 4 = [ 0, 0, 0, 0, 0, 0, 0, 0 ]
Cohomology in dimension 5 = [ 0, 0, 0 ]
Cohomology in dimension 6 = [ 0 ]
gap> Dimension(R)(7);
0
gap> List([1..m-1],n->Length(IntegralRingGenerators(R,n)));
[ 3, 8, 6, 0, 0, 0]
```

□

Theorem 2.7(i) was originally proved in [**15**]. Theorem 2.7(ii) appears to be new.

Theorem 2.8. [**16**] *The finite-type Artin group A corresponding to the exceptional finite reflection group F_4 has integral cohomology*

$$H^1(A,\mathbb{Z}) \cong \mathbb{Z}^2, \qquad H^2(A,\mathbb{Z}) \cong \mathbb{Z}^2, \qquad H^3(A,\mathbb{Z}) \cong \mathbb{Z}^2,$$
$$H^4(A,\mathbb{Z}) \cong \mathbb{Z}, \qquad H^n(A,\mathbb{Z}) \cong 0\,(n \geq 5).$$

Proof.

```
gap> D:=[[1,[2,3]],[2,[3,4]],[3,[4,3]]];;
gap> Display(List([1..5],i->GroupCohomology(D,i)));
 [ [ 0, 0 ], [ 0, 0 ], [ 0, 0 ], [ 0 ], [  ] ]
```

Explanation of proof. An Artin group is represented in HAP by a list D which describes the group's Coxeter diagram. The Coxeter diagram for F_4 has four vertices and three labelled edges. The first and second vertices are connected by an edge labelled by 3; the second and third vertices are connected by an edge labelled by 4, the third and final vertices are connected by an edge labelled 3. The HAP function `GroupCohomology()` can be applied directly to D. □

Theorem 2.8 was originally proved in [**16**] where the cohomology of each of the seven exceptional finite-type Artin groups was calculated.

Theorem 2.9. [**1, 8**] *The 3-generator affine braid group $A = \langle x, y, z : xyx = yxy, yzy = zyz, xzx = zxz\rangle$ admits a 2-dimensional classifying space, namely the CW-space associated to its presentation.*

Proof. We must show that the 2-dimensional CW-space B associated to the presentation is aspherical. Asphericity is obviously a homotopy invariant. So we can alternatively test the asphericity of the homotopy equivalent space B' arising from the presentation of A obtained by adding generators a, b, c, adding relations $a = xy$, $b = yz$, $c = zx$ and replacing all occurrences of xy, yz, and zx.

```
gap> F:=FreeGroup(6);;
gap> x:=F.1;;y:=F.2;;z:=F.3;;a:=F.4;;b:=F.5;;c:=F.6;;
gap> rels:=[a^-1*x*y, b^-1*y*z, c^-1*z*x, a*x*(y*a)^-1,
> b*y*(z*b)^-1, c*z*(x*c)^-1];;
```

```
gap> IsAspherical(F,rels);
Presentation is aspherical.
true
```

□

Theorem 2.9 is a particular example of a result of Appel and Schupp [**1**] who proved that all Artin groups of large type have 2-dimensional classifying spaces. Theorem 2.9 is also a special case of a result of Charney and Peiffer [**8**] who showed that the $(n+1)$-generator affine braid group admits an n-dimensional classifying space.

PROPOSITION 2.10. *The mod 2 cohomology ring $H^*(D_{64}, \mathbb{Z}_2)$ of the dihedral group of order 64 is generated by two elements in degree 1, one element in degree 2, and possibly (though not very likely) some generators of degree greater than 20.*

PROOF.
```
gap> A:=ModPCohomologyRing(DihedralGroup(64),20);
gap> List(ModPRingGenerators(A),a->A!.degree(a));
[ 0, 1, 1, 2 ]
```

□

PROPOSITION 2.11. *The amalgamated free product $S_5 *_{S_3} S_4$ of the symmetric groups S_5 and S_4 over S_3 has seventh integral homology $H_7(S_5 *_{S_3} S_4, \mathbb{Z}) = (Z_2)^3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{60}$.*

PROOF.
```
gap> S5:=SymmetricGroup(5);; S4:=SymmetricGroup(4);;
gap> S3:=SymmetricGroup(3);;
gap> S3S5:=GroupHomomorphismByFunction(S3,S5,x->x);;
gap> S3S4:=GroupHomomorphismByFunction(S3,S4,x->x);;
gap> D:=[S5,S4,[S3S5,S3S4]];;
gap> R:=ResolutionGraphOfGroups(D,8);;
gap> Homology(TensorWithIntegers(R),7);
[ 2, 2, 2, 4, 60 ]
```

Explanation of proof. An amalgamated free product is a special case of a graph of groups. The function `ResolutionGraphOfGroups(D,n)` inputs a list D describing a graph of groups, together with a positive integer n. It returns n terms of a free $\mathbb{Z}G$-resolution for the group G described by the graph of groups. □

PROPOSITION 2.12. *Let $G = \mathrm{GL}_4(3)$ be the group of invertible 4×4 matrices over the field of three elements; this is a non-perfect, non-solvable group of order 24261120. Let $SK(G,1)$ be the suspension of a classifying space for G. Then this suspension space has third homotopy group $\pi_3(SK(G,1)) = \mathbb{Z}_2$.*

PROOF. The following commands are based on the isomorphism

$$\pi_3(SK(G,1)) = \ker(G \otimes G \to G),$$

involving the nonabelian tensor square [**5**].

```
gap> G:=Image(IsomorphismPermGroup(GL(4,3)));;
gap> ThirdHomotopyGroupOfSuspensionB_alt(G);
[ [  ], [ 2 ] ]
```

□

3. The role of contracting homotopies

A free kG-resolution R_* is represented in HAP as a record R with various components such as `R!.boundary(n,i)` which gives the image of the ith free generator e_i^n of R_n under the boundary map $d_n\colon R_n \to R_{n-1}$, and `R!.group` which gives the group G. One of the less obvious components `R!.homotopy(n,[i,g])` returns, for $g \in G$, the image of $g \cdot e_i^n$ under a contracting homotopy $h_n\colon R_n \to R_{n+1}$.

Recall that a contracting homotopy on R_* is a family of abelian group homomorphisms $h_n\colon R_n \to R_{n+1}$ $(n \geq 0)$ satisfying $d_{n+1}h_n(x) + h_{n-1}d_n(x) = x$ for all $x \in R_n$ (where $h_{-1} = 0$). Since the h_n are not G-equivariant one needs to specify $h_n(x)$ on a set of abelian group generators for R_n. Having specified a contracting homotopy, one can use it to make constructive the following element of choice which frequently occurs in homological algebra:

> *For each $x \in \ker(d_n\colon R_n \to R_{n-1})$ choose an element $\tilde{x} \in R_{n+1}$ such that $d_{n+1}(\tilde{x}) = x$.*

The choice can be made by setting $\tilde{x} = h_n(x)$.

Thus contracting homotopies are used in HAP for: (i) constructing homology homomorphisms $H_*(G,N) \to H_*(G',N')$ induced by group homomorphisms $\phi\colon G \to G'$ and ϕ-equivariant module homomorphisms $N \to N'$; (ii) constructing cup products $H^p(G,k) \otimes H^q(G,k) \to H^{p+q}(G,k)$; (iii) applying perturbation techniques which input free kH-resolutions for various subgroups H in G and output a free kG-resolution.

4. Linear algebra techniques

For a small group G and integral domain k (typically $k = \mathbb{Z}$ or k a finite field) one can naively represent an element in the group ring kG as a vector of length $|G|$ over k. An element in a free kG-module $(kG)^t$ can be represented as a vector of length $t \times |G|$ over k. To compute a free kG-resolution

$$R_*\colon \quad \cdots \to R_n \to R_{n-1} \to \cdots \to R_0,$$

one can set $R_0 = kG$, define $d_0\colon kG \to k$, $\Sigma\lambda_g g \mapsto \Sigma\lambda_g$, and then recursively

(1) determine $\ker d_n$ (using gaussian elimination if k is a field, and Smith Normal Form if $k = \mathbb{Z}$),
(2) determine a small subset $\{v_1, \ldots, v_t\} \subset \ker d_n$ whose kG-span equals $\ker d_n$,
(3) set $R_{n+1} = (kG)^t$,
(4) define $d_{n+1}\colon (kG)^t \to R_n$ by sending the ith free generator to v_i.

The construction of a contracting homotopy $h_n\colon R_n \to R_{n+1}, x \mapsto \tilde{x}$ essentially boils down to solving a matrix equation $d_{n+1}(\tilde{x}) = x$ where $\tilde{x}$ is unkown and x is a known vector in the image of d_{n+1}. The most costly part of the recursive procedure is Step 2. If k is the field of p elements and G a p-group then the radical of $\ker d_n$ is the vector space spanned by vectors $v - g \cdot v$ where v ranges over a k-basis for

$\ker d_n$ and g ranges over generators for G. Any basis for the complement of the radical yields a minimal set $\{v_1, \cdots v_t\}$ with kG-span equal to $\ker d_n$. Even when G is not a p-group this method can be used to find a set whose kP-span equals $\ker d_n$, where P is a Sylow p-subgroup of G. Given a non-minimal set with kG-span equal to $\ker d_n$ one can use naive methods to find a minimal subset, even in the case where $k = \mathbb{Z}$.

These linear algebraic techniques are used in the proofs of Theorems 2.3, 2.4 and Proposition 2.10.

5. Geometric techniques

Naive linear algebra can obviously only be applied to extremely small groups. For larger groups, or for integral coefficients, one can make use of the following fact from elementary algebraic topology.

> If a group G acts fixed-point freely and cellularly on a contractible CW-space$X$ then the cellular chain complex
>
> $$C_*(X) : \cdots \to C_2(X) \to C_1(X) \to C_0(X)$$
>
> is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

The quotient X/G of such a space X obtained by killing the action of G is a *classifying space* for the group G. Recall that $C_n(X)$ is the free abelian group whose free generators correspond to the n-cells of X. The boundary homomorphism $d_n : C_n(X) \to C_{n-1}(X)$ can be obtained directly from an explicit description of X.

For a finite group G one can recursively construct the n-skeleta

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$$

of a suitable CW-space X by:

(1) setting $X^0 = G$,
(2) constructing X^{n+1} by attaching just enough $(n+1)$-cells to X^n so that
  (a) $\pi_n(X^{n+1}) = 0$,
  (b) G freely permutes the cells of X^{n+1}.

This technique is used in the proofs of Theorems 2.1, 2.2, 2.6 and Proposition 2.11.

One can also use Polymake computational geometry software to obtain a contractible space X with free G-action. For example, suppose we have a faithful linear representation $\alpha : G \to \mathrm{GL}_n(\mathbb{R})$ of a finite group G. We can choose a vector $v \in \mathbb{R}^n$ with trivial stabilizer group in G and define the *orbit polytope* $P(G)$ as

$$P(G) = \text{Convex hull}\{\alpha g(v) : g \in G\}.$$

The polytope $P(Q)$ is a contractible m-dimensional CW-space ($m \le n$) and G acts on it by permuting cells. Consequently its cellular chain complex $C_* = C_*(P(G))$ is an exact sequence of $\mathbb{Z}G$-modules with $H_0(C_*) = \mathbb{Z}$ and $C_m = \mathbb{Z}$. Copies of C_* can be spliced together to form a periodic $\mathbb{Z}G$-resolution

$$\cdots \to C_1 \to C_0 \to C_{m-1} \to \cdots \to C_1 \to C_0 \to C_{m-1} \to \cdots \to C_1 \to C_0$$

of period m. If G acts freely on the $(m-1)$-skeleton of $P(G)$ then this is a free periodic resolution. This polytopal resolution is used in the proof of Theorem 2.5 where the relevant orbit polytope is computed using Polymake software [**12**].

If G does not act freely on the $(m-1)$-skeleton of $P(G)$ then the perturbation techniques described in the next section can be used to convert $C_*(P(G))$ into a free resolution.

Polymake software can also be used to construct resolutions for infinite groups such as Bieberbach groups G. Here one uses convex hull computations to construct a Dirichlet-Voronoi fundamental domain for the action of G on Euclidean space $\mathbb{R}^n$. This yields a tessellation (and hence CW-structure) on the contractible space $\mathbb{R}^n$. The resulting cellular chain complex $C_*(\mathbb{R}^n)$ is a $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

A third use of Polymake is made in the proof of Theorem 2.9. The idea here is to show that a certain space B can not contain a 2-sphere. An argument based on Euler characteristics is encoded as a linear programming problem and then solved using Polymake.

A theoretically constructed polytopal classifying space can be used to compute the cohomology of certain Artin groups. To explain this we recall that a *Coxeter matrix* is a symmetric $n\times n$ matrix each of whose entries $m(i,j)$ is a positive integer or ∞, with $m(i,j)=1$ if and only if $i=j$. Such a matrix can be represented by the *Coxeter graph* D with n vertices, and with a labelled edge joining vertices i and j if $m(i,j)\geq 3$. An *Artin group* A_D and *Coxeter group* W_D is assigned to each Coxeter graph as follows. The Artin group A_D is generated by a set of elements $S=\{x_1,\cdots x_n\}$ subject to the relations $(x_ix_j)_{m(i,j)}=(x_jx_i)_{m(i,j)}$ for all $i\neq j$, where $(xy)_m$ denotes the word $xyxyx\cdots$ of length m. The Coxeter group W_D is the group satisfying the additional relations $x^2=1$ for $x\in S$. Denote by $\overline{x}$ the image in W_D of the generator x.

A finite Coxeter group W_D can be realized as a group of orthogonal transformations of $\mathbb{R}^n$ with generators $\overline{x}$ equal to reflections [**14**]. For any point v in $\mathbb{R}^n$ lying in the complement of the generating mirrors we denote by P_D the convex hull of the orbit of v under the action of W_D. It is readily shown that the face lattice of the n-dimensional convex polytope P_D depends only on the graph D, and that the polytope is simple (i.e. each vertex touches precisely n edges). Simplicity implies that the face lattice of the polytope is in fact determined by its 1-skeleton [**2**]. Label each edge in P_D by the unique generator $\overline{x}$ stabilizing it. Define the *length* of an element g in W_D to be the shortest length of a word in the generators representing it. It is possible to orient each edge in P_D so that its initial vertex gv and final vertex $g'v$ are such that the length of g is less than the length of g'. With this orientation the boundaries of 2-faces in P_D spell words corresponding to the relators of the Artin group A_D.

For an arbitrary Coxeter graph D we define a CW-space B_D as follows. For each full subgraph D_i in D, that is maximal with respect to the property that W_{D_i} is a finite subgroup, we construct the oriented and labelled convex polytope W_{D_i}. The space B_D is obtained from the union of the polytopes W_{D_i} by identifying in a canonical fashion all faces with the same labelled and directed edge graph. It is conjectured that B_D is a classifying space for the Artin group A_D [**7**]. In cases where this conjecture is known, the cellular chain complex $C_*(\tilde{B}_D)$ of the universal cover of B_D is a free $\mathbb{Z}A_D$-resolution. This free resolution is used, for example, in our proof of Theorem 2.8.

6. Using subgroups

We mention two ways in which subgroups of G can be brought into the calculation of the (co)homology of G.

The first technique concerns a finite group G with Sylow p-subgroup P. There is a surjection $H_n(P,\mathbb{Z}) \to H_n(G,\mathbb{Z})_{(p)}$ from the homology of P onto the p-part of the homology of G. The kernel of this surjection is described in terms of double coset representatives

$$G = \bigcup_x PxP$$

and induced homomorphisms $H_n(P,\mathbb{Z}) \to H_n(xPx^{-1},\mathbb{Z})$. by the classical Cartan-Eilenberg double coset formula. Thus, the homology of a large finite group can be deduced from that of its Sylow subgroups. This technique is used, for example, in our proof of Theorems 2.1 and 2.3.

The second technique concerns a group G which may be infinite. Suppose we have a $\mathbb{Z}G$-resolution of $\mathbb{Z}$

$$C_*\colon\ \cdots \to C_n \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z}.$$

but that C_* is not free. Suppose, however, that for each m we have a free $\mathbb{Z}G$-resolution of the module C_m

$$D_{m*}\colon\ \to D_{m,n} \to D_{m,n-1} \to \cdots \to D_{m,0} \to C_m.$$

Theorem 6.1. [**24**] *There is a free* $\mathbf{Z}G$*-resolution* $R_* \to \mathbb{Z}$ *with*

$$R_n = \bigoplus_{p+q=n} D_{p,q}$$

The proof of this Theorem given in [**24**] can be made constructive by using contracting homotopies on the resolutions D_{m*}. Furthermore, a contracting homotopy on R_* can be constructed by a formula involving contracting homotopies on the D_{m*} and on C_*.

The following three scenarios are covered by Theorem 6.1:

(1) C_* is the cellular chain complex of an orbit polytope;
(2) C_* is the cellular chain complex of a graph of groups;
(3) C_* is a free $\mathbb{Z}(G/N)$-resolution.

Theorem 6.1 is used for example in our proofs of Theorem 2.2 and Proposition 2.11.

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