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Computing covers of Lie algebras

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Abstract. We describe an algorithm for computing the Lie cover and multiplier of a finite dimensional Lie algebra over a field. A variant of the algorithm can, in certain cases, be used to compute the Leibniz cover of a Lie algebra. The algorithm has been implemented in GAP and the paper presents some computer examples.

1. Introduction

Let L be a finite dimensional Lie algebra over a field k. A *stem extension* of L is a short exact sequence of Lie algebras

$$0 \to M \to C \xrightarrow{\phi} L \to 0$$

in which M lies in both the centre $Z(C)$ and derived subalgebra C^2. A stem extension is called a *Lie cover* when the vector space M has dimension equal to that of the Chevalley-Eilenberg homology $H_2(L,k)$. One purpose of this paper is to describe an algorithm for computing the covering homomorphism ϕ; the algorithm provides a new proof of the known fact [6] that Lie covers exist and are unique up to isomorphism. The algorithm has been implemented as part of the HAP package for the GAP computational algebra system. We illustrate this implementation on a solvable (but non-nilpotent) rational Lie algebra introduced in [8].

We should note that the GAP package SOPHUS already contains an efficient implementation of an alternative algorithm for computing covers of nilpotent Lie algebras over finite fields; this is based on the methods in [7] for covering groups of finite p-groups.

The Lie cover of L can be used to decide if there exists some Lie algebra K whose central quotient $K/Z(K)$ is isomorphic to L. Such a K exists if and only if a certain central ideal $Z^*(L)$, which we term the *epicentre* of L, is trivial. The Lie cover has another useful property, namely that all stem extensions of L arise as the quotient of the cover by an ideal in the multiplier. Proofs of these properties were given in [1]; they also follow directly from our algorithm.

A second purpose of the paper is to describe how our method can, in certain cases, be used to compute the *Leibniz cover* of a Lie algebra L. This cover is

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analogous to the Lie cover except that C is only required to be a Leibniz algebra and the central ideal M is required to have the same dimension as that of the Leibniz homology $HL_2(L,k)$. We illustrate this on the Lie algebra of 4×4 rational matrices.

2. Nonabelian tensor squares

For convenience we shall use the language of nonabelian tensor products of Lie algebras introduced in [2, 3], though we specialze to the very particular case of the tensor product of a Lie algebra with itself.

Let L and T be Lie algebras over the field k. A bilinear function $\tau\colon L\times L\to T$ is said to be a *Lie pairing* if

$$\begin{aligned}\tau([x,x'],y) &= \tau(x,[x',y])-\tau(x',[x,y]),\\ \tau(x,[y,y']) &= \tau([y',x],y)-\tau([y,x],y'),\\ \tau([y,x],[x',y']) &= -[\tau(x,y),\tau(x',y')],\end{aligned}$$

for all $x,x',y,y'\in L$. The *nonabelian tensor square* $L\otimes L$ arises as a Lie pairing $\otimes\colon L\times L\to L\otimes L$ and is defined up to isomorphism by the following universal property: for any Lie pairing $\tau\colon L\times L\to T$ there exists a unique Lie homomorphism $\psi\colon L\otimes L\to T$ making the following diagram commute.

$$\begin{array}{ccc} L\times L & \xrightarrow{\ \otimes\ } & L\otimes L \\ & \searrow^{\tau} & \downarrow{\scriptstyle\psi} \\ & & T \end{array}$$

The commutator pairing $L\times L\to L,(x,y)\mapsto[x,y]$ induces the homomorphism $\partial\colon L\otimes L\to L$. We denote by $J_2(L)$ the kernel of ∂. One can show [3] that $J_2(L)$ is a central ideal in $L\otimes L$.

Since Lie pairings are bilinear the vector space underlying the nonabelian tensor square is a quotient of the usual tensor product $L\otimes_k L$. The following lemma (whose proof is left to the reader) provides a constructive description of the kernel of this quotient.

Lemma 1. *Let B be a k-basis for the finite-dimensional Lie algebra L. Let I be the vector subspace of the vector space tensor product $L\otimes_k L$ spanned by the vectors*

$$([x,x']\otimes y)-(x\otimes[x',y])+(x'\otimes[x,y]),$$
$$(x\otimes[y,y'])-([y',x]\otimes y)+([y,x]\otimes y')$$

for $x,x',y,y'\in B$. Let $\mu\colon(L\otimes_k L)\times(L\otimes_k L)\to L\otimes_k L$ be the bilinear map defined on basis elements by $\mu((x\otimes y),(x'\otimes y'))\mapsto -[y,x]\otimes[x',y']$. Then μ induces the structure of a Lie algebra on the quotient vector space $(L\otimes_k L)/I$ and there is a Lie isomorphism $L\otimes L\cong(L\otimes_k L)/I$.

Lemma 1 has been implemented as part of the HAP homological algebra package [4] as a computer function `LieTensorSquare(L)` which inputs a finite-dimensional Lie algebra L represented as a structure constants algebra. It returns the homomorphism $L\otimes L\to L$ as a homomorphism of structure constants algebras; it also returns the Lie pairing function $\otimes\colon L\times L\to L\otimes L$.

A Lie pairing $\tau\colon L\times L\to T$ is said to be an *exterior* Lie pairing if

$$\tau(x,x)=0$$

for all $x \in L$. The *nonabelian exterior square* $L \wedge L$ is the Lie algebra arising in a universal exterior pairing $\tau\colon L \times L \to L \wedge L$. The commutator pairing

$$\begin{array}{ccc} L \times L & \to & L \\ (x,y) & \mapsto & [x,y] \end{array}$$

again induces a Lie homomorphism $\delta\colon L \wedge L \to L$ whose kernel we denote by $H_2(L)$. This kernel is a central ideal in $L \wedge L$. It was observed in [2] that $H_2(L)$ is isomorphic to the second Chevalley-Eilenberg homology $H_2(L,k)$.

The proof of the following lemma is left to the reader.

Lemma 2. *Let B be a k-basis for the finite-dimensional Lie algebra L. Let J be the vector subspace of the nonabelian tensor square $L \otimes L$ spanned by the vectors*

$$x \otimes x,$$

$$x \otimes y + y \otimes x = 0$$

for $x, y \in B$. Then J is an ideal and there is a Lie isomorphism $L \wedge L \cong (L \otimes L)/J$.

The computer function `LieExteriorSquare(L)` of [4] implements Lemma 2; the function can be applied to any finite-dimensional Lie algebra L represented as a structure constants algebra.

Example 1. Consider the Lie algebra $L = M_4(\mathbb{Q})$ of 4×4 rational matrices. The following HAP commands show that the second Chevalley-Eilenberg homology $H_2(L,\mathbb{Q})$ is trivial, the second Leibniz homology $HL_2(L,\mathbb{Q})$ has dimension 1, and $J_2(L)$ is isomorphic to $HL_2(L,\mathbb{Q})$. The homology is calculated from implementations in HAP of the standard Chevalley-Eilenberg complex and the standard Leibniz complex. See [5] for a description of these two complexes.

```
gap> L:=MatLieAlgebra(Rationals,4);;
gap> LieAlgebraHomology(L,2);
0
gap> LeibnizAlgebraHomology(L,2);
1
gap> J:=Kernel(LieTensorSquare(L).homomorphism);;
gap> Dimension(J);
1
```

3. A construction of the Lie cover

In a stem extension of Lie algebras

$$0 \to M \to C \xrightarrow{\phi} L \to 0 \qquad (1)$$

the Lie algebra C is uniquely determined, up to isomorphism, by (i) the derived subalebra C^2, (ii) the restricted homomorphism $\delta = \phi|_{C^2}\colon C^2 \to L$, and (iii) the exterior pairing $\tau\colon L \times L \to C^2$, $(x,y) \mapsto [\tilde{x},\tilde{y}]$, where $\tilde{x} \in C$ denotes an arbitrary lift of $x \in L$. Since M is central the pairing τ is well-defined.

To recover C from the three pieces of data we first construct: a vector space V_{C^2} of dimension $\dim(C^2)$, a vector space $V_{L^{\rm ab}}$ of dimension $\dim(L^{\rm ab})$, the direct sum of vector spaces $V = V_{C^2} \oplus V_{L^{\rm ab}}$, an inclusion of vector spaces $\iota\colon C^2 \to V$ mapping C^2 onto the summand V_{C^2}, and an inclusion of vector spaces $\sigma\colon V_{L^{\rm ab}} \to L$ whose image has trivial intersection with the derived subalgebra L^2. The vector space V

has dimension equal to the dimension of C since $C^{\mathrm{ab}} = L^{\mathrm{ab}}$. There is a linear homomorphism

$$\psi\colon V = V_{C^2} \oplus V_{L^{\mathrm{ab}}} \to L, \quad (x, y) \mapsto \delta(\iota^{-1}(x)) + \sigma(y)$$

and this homomorphism can be composed with the exterior pairing τ and inclusion ι to produce a bilinear bracket operation on V defined by

$$[\ ,\ ]\colon V \times V \to V, \quad (v, w) \mapsto \iota(\tau(\psi(v), \psi(w))).$$

The proof of the following lemma is left to the reader.

Lemma 3. *The bracket on V is a Lie bracket and the Lie algebra V is Lie isomorphic to C. Modulo this isomorphism the homomorphism ψ corresponds to the stem extension homomorphism ϕ.*

For any finite dimension Lie algebra L the methods of Section 2 can be used to compute the following three pieces of data: (i) the nonabelian exterior square $L \wedge L$, (ii) the Lie homomorphism $\delta\colon L \wedge L \to L$, $(x, y) \mapsto [x, y]$, and (iii) the universal exterior pairing $\wedge\colon L \times L \to L \wedge L$. Since $\ker(\delta) \cong H_2(L, k)$ we can apply the construction of Lemma 3 to this starting data in order to obtain a Lie covering

$$0 \to H_2(L, k) \to L^* \to L \to 0$$

in which the derived subalgebra of L^* is isomorphic to the exterior square $L \wedge L$.

Since the exterior square is uniquely determined by L up to isomorphism we obtain the following.

Theorem 4. [6] *Any finite dimensional Lie algebra L admits a covering homomorphism $\phi\colon L^* \to L$ and L^* is unique up to Lie isomorphism.*

Theorem 4 is in contrast to the group theoretic situation where finite groups generally admit non-isomorphic covers.

For an arbitrary stem extension (1) the exterior pairing

$$\begin{array}{rcl} \tau\colon L \times L & \longrightarrow & C^2, \\ (x, y) & \longmapsto & [\tilde{x}, \tilde{y}] \end{array}$$

induces a surjection $L \wedge L \to C^2$ whose kernel lies in $\ker(\delta\colon L \wedge L \to L) \cong H_2(L, k)$. We thus obtain the following result.

Theorem 5. [1] *In any stem extension (1) the Lie algebra C is isomorphic to L^*/I where I is a subspace of the vector space $H_2(L, k)$. Moreover, any subspace $I \leq H_2(L, k)$ gives rise to a stem extension.*

The above construction of the Lie cover has been implemented as a function `LieCoveringHomomorphism(L)` in the HAP package.

Example 2. Let L be a 13-dimensional vector space over $\mathbb{Q}$ with basis

$$\{u, x, y, z, t, e_1, e_2, f_1, f_2, g_1, g_2, h_1, h_2\}.$$

It was observed in [8] that a solvavble Lie algebra structure can be imposed on L by defining a bilinear bracket with

$$\begin{array}{rlrll}
[u,a_2] &= -a_1, & [u,a_1] &= a_2, & \text{for } a=e,f,g,h,\\
[e_1,e_2] &= x, & [f_1,f_2] &= y, & \\
[e_1,f_2] &= -t, & [e_2,f_1] &= t, & \\
[x,f_i] &= h_i, & [y,e_i] &= g_i, & \text{for } i=1,2,\\
[t,e_i] &= -h_I, & [t,f_i] &= -g_i, & \text{for } i=1,2,\\
[e_1,g_2] &= -\frac{1}{2}z, & [e_2,g_1] &= \frac{1}{2}z, & \\
[f_1,h_2] &= \frac{1}{2}z, & [f_2,h_1] &= -\frac{1}{2}z, & \\
[x,y] &= z, & & &
\end{array}$$

and with all other products of generators zero. This Lie algebra can be entered into GAP as a structure constants algebra by using commands

```
gap> SCTL:=EmptySCTable(13,0,"antisymmetric");;
gap> SetEntrySCTable( SCTL, 1, 6, [ 1, 7 ] );;
gap> SetEntrySCTable( SCTL, 1, 8, [ 1, 9 ] );;

     ⋮

gap> L:=LieAlgebraByStructureConstants(Rationals,SCTL);;
```

The GAP commands

```
gap> IsLieSolvable(L);
true
gap> IsLieNilpotent(L);
false
```

show that L is solvable but not nilpotent. The following HAP commands show that the Lie cover L^* is 15-dimensional, has derived length equal to the derived length of L, has trivial Lie multiplier $H_2(L^*,2)=0$, and has 6-dimensional second Leibniz homology $HL_2(L^*,k)=k^6$. They also show that L is not isomorphic to a central quotient of Lie algebras $K/Z(K)$.

```
gap> phi:=LieCoveringHomomorphism(L);;
gap> Lstar:=Source(phi);;
gap> Dimension(Lstar);
15
gap> Length(LieDerivedSeries(Lstar))=Length(LieDerivedSeries(L));
true
gap> LieAlgebraHomology(Lstar,2);
0
gap> LeibnizAlgebraHomology(Lstar);
6
gap> LieAlgebraHomology(Lstar,2);
0
gap> Dimension(LieEpicentre(L))>0;
true
```

4. Leibniz algebras

A *Leibniz algebra* is a vector space G over k with a bilinear bracket

$$[\ ,\ ] \colon G \times G \longrightarrow G$$

satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for all $x, y, z \in G$. Any Lie algebra is a Leibniz algebra. Conversely, a Leibniz algebra is a Lie algebra if $[x, x] = 0$ for all $x \in G$.

The homology $HL_*(G, k)$ of a Leibniz algebra is the homology of the chain complex

$$\cdots \to G^{\otimes n} \xrightarrow{d} G^{\otimes n-1} \to \cdots \to G \to k$$

with boundary map defined by the formula

$$d(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \le i < j \le n} (-1)^j (x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n).$$

See for example [5].

Given a stem extension $HL_2(G, k) \to G^{\#} \xrightarrow{\phi} G$ of Leibniz algebras we say that $G^{\#}$ is a *Leibniz cover* of G.

Consider now a Lie algebra L. If in our construction of the Lie cover L^* we were to replace the exterior square $L \wedge L$ with the nonabelian tensor square $L \otimes L$ we would get a stem extension

$$0 \to J_2(L) \to C \to L \to 0$$

where C is a Leibniz algebra with derived subalgebra $C^2 \cong L \otimes L$. If $J_2(L)$ happens to equal $HL_2(L, k)$ then C is a Leibniz cover of L.

Example 3. Consider the Lie algebra $L = M_4(\mathbb{Q})$ of 4×4 rational matrices. We saw in Example 1 that $J_2(L) = HL_2(L, k)$. The following HAP commands use the above technique to calculate a Leibniz cover $L^{\#}$ for L and show that $HL_2(L^{\#}, \mathbb{Q}) \cong \mathbb{Q}$.

```
gap> L:=MatLieAlgebra(Rationals,4);;
gap> Lcover:=Source(LeibnizQuasiCoveringHomomorphism(L));;
gap> LeibnizAlgebraHomology(L,1);
1
gap> LeibnizAlgebraHomology(C,2);
1
```

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