ON THE INTEGRAL HOMOLOGY OF $\mathrm{PSL}_4(\mathbb{Z})$ AND OTHER ARITHMETIC GROUPS

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Abstract. We determine the integral homology of $\mathrm{PSL}_4(\mathbb{Z})$ in degrees up to 5 and determine its p-part in higher degrees for the primes $p \geq 5$. Our method applies to other arithmetic groups; as illustrations we include descriptions of the integral homology of $\mathrm{PGL}_3(\mathbb{Z}[i])$ and $\mathrm{PGL}_3(\mathbb{Z}[\exp(2\pi i/3)])$ in degrees up to 5.

1. Introduction

We obtain the following partial description of the integral homology of the projective special linear group $\mathrm{PSL}_4(\mathbb{Z})$ (writing $A_{(p)}$ to denote the p-primary part of an Abelian group A).

Theorem 1.

(i) $$H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z}) = \begin{cases} 0 & n = 1 \\ (\mathbb{Z}_2)^3 & n = 2 \\ \mathbb{Z} \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_3)^2 \oplus \mathbb{Z}_5, & n = 3 \\ (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_5, & n = 4 \\ (\mathbb{Z}_2)^{13}, & n = 5. \end{cases}$$

(ii) $$H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)} = \begin{cases} \mathbb{Z}_5 & n \equiv 0, 3 \bmod 4 \ (n \geq 6) \\ 0 & n \equiv 1, 2 \bmod 4 \ (n \geq 6) \end{cases}$$

(iii) $H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(p)} = 0$ *for all primes* $p \geq 7$ *and* $n \geq 0$.

(iv) $H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})$ *is finite for all positive* $n \neq 3$.

The proof of Theorem 1(i) involves four steps. The first, described in Section 2, uses computer calculations of perfect forms to explicitly determine a CW-structure on a homotopy retract X of the space $S^4_{>0}$ of 4×4 positive definite symmetric matrices. The retract X is 6-dimensional, contractible and admits a cellular action of $G = \mathrm{PSL}_4(\mathbb{Z})$ in which each cell e has a finite stabilizer group G^e. (We remark that an analogous 3-dimensional retract of $S^3_{>0}$, due to Avner Ash, was used by Christophe Soulé [19] in his calculation of the integral cohomology of $\mathrm{SL}_3(\mathbb{Z})$. Furthermore, versions of our retract X for $\mathrm{PSL}_4(\mathbb{Z})$ have been determined by Ash et al. in [2] and by Elbaz-Vincent et al. in [7, 6], who used it to compute cohomology rationally and integrally at large primes only.) We will refer

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to [16] and [12] for full details on perfect forms and limit our exposition to those facts necessary for Theorem 1.

The next step is described in Section 3 and uses algorithms from [8, 10] to determine small free $\mathbb{Z}G^e$-resolutions of $\mathbb{Z}$ for each stabilizer group. Implementations of these algorithms are available in the HAP [9] software package. However, Theorem 1 was obtained using an implementation specially adapted by the first author.

The third step is described in Section 4. It uses a generalization of a lemma of C.T.C. Wall to combine the resolutions for the stabilizer groups with the cellular chain complex $C_*(X)$ to form a free $\mathbb{Z}G$-resolution R_* of $\mathbb{Z}$. (For a full exposition of this technique we will refer to [11] where it was illustrated with the detailed hand calculation of free resolutions for generalized triangle groups acting on hyperbolic 3-space.) The technique has been automated in the software package [9] and such an automation was used to obtain six terms of a free $\mathbb{Z}G$-resolution R_*.

The final step in the proof of Theorem 1(i) is routine: the Smith Normal Form algorithm is used to compute the homology of the chain complex $R_* \otimes_{\mathbb{Z}G} \mathbb{Z}$.

Theorem 1(ii)-(iv) are standard applications of the Leray spectral sequence. Details are given in Section 5. The following result, which gives bounds on the 3-primary part of the homology, is also derived in Section 5.

Proposition 2. *At the prime* 3 *the Leray spectral sequence*

$$E^1_{p,q} = \bigoplus_{[e^p]} H_q(G^{e^p}, \mathbb{Z}^\rho)_{(3)} \Rightarrow H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(3)}$$

(where $[e^p]$ *ranges over the* G*-orbits of* p*-dimensional* e^p *cells in* X *and* ρ *is a suitable action of* G^{e^p} *on* $\mathbb{Z}$*) has first page*

$$E^1_{p,q} = \begin{cases} (\mathbb{Z}_3)^{q/2} & p=0, q=2+4k \\ (\mathbb{Z}_3)^{(5+q)/2} & p=0, q=3+4k \\ (\mathbb{Z}_3)^{(q+2)/4} & p=1, q=2+4k \\ (\mathbb{Z}_3)^{(q+5)/4} & p=1, q=3+4k \\ \mathbb{Z}_3 & p=3, q=1+2k \\ \mathbb{Z}_3 & p=4, q=1+4k \\ (\mathbb{Z}_3)^{(q+2)/4} & p=4, q=2+4k \\ (\mathbb{Z}_3)^{(q+5)/4} & p=4, q=3+4k \\ \mathbb{Z}_3 & p=5, q=1+2k \\ \mathbb{Z}_3 & p=6, q=1+4k \\ 0 & \text{otherwise} \end{cases} \qquad (k \geq 0).$$

The techniques underlying Theorem 1 can in principle be applied to other arithmetic groups. For example, they yield the following descriptions of the low-dimensional integral homology of the 3-dimensional projective general linear groups $\mathrm{PGL}_3(\mathbb{Z}[i])$, $\mathrm{PGL}_3(\mathbb{Z}[\omega])$ over the Gaussian and Eisenstein integers (where $\omega = \exp(2\pi i/3)$).

Theorem 3.

$$H_n(\mathrm{PGL}_3(\mathbb{Z}[i]),\mathbb{Z}) = \begin{cases} 0 & n=1 \\ (\mathbb{Z}_2)^2 & n=2 \\ \mathbb{Z}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_8\oplus\mathbb{Z}_3, & n=3 \\ \mathbb{Z}\oplus\mathbb{Z}_2, & n=4 \\ (\mathbb{Z}_2)^5\oplus(\mathbb{Z}_4)^2, & n=5. \end{cases}$$

$$H_n(\mathrm{PGL}_3(\mathbb{Z}[\omega]),\mathbb{Z}) = \begin{cases} \mathbb{Z}_3 & n=1 \\ \mathbb{Z}_3 & n=2 \\ \mathbb{Z}\oplus\mathbb{Z}_8\oplus(\mathbb{Z}_3)^3, & n=3 \\ \mathbb{Z}\oplus(\mathbb{Z}_3)^2, & n=4 \\ (\mathbb{Z}_2)^2\oplus(\mathbb{Z}_3)^3, & n=5. \end{cases}$$

The method of Theorem 1(i) and Theorem 3 has also been used to obtain the first few terms of an explicit resolution for the symplectic group $\mathrm{Sp}_4(\mathbb{Z})$. Although the integral (co)homology has previously been calculated in this case by other means (see [5]), an advantage of an explicit resolution is that in principle it can be used to compute the cohomology and explicit cocycles for finite index subgroups such as congruence subgroups. See the final section for details.

We will not give details for the proof of Theorem 3. Instead, in Section 6 we explain how the results of the theorem can be computed using the HAP [9] software package. (We remark that the integral homology of $\mathrm{PSL}_2(\mathbb{O})$ for various rings of quadratic integers $\mathbb{O}$ have been calculated by J. Schwermer and K. Vogtmann in [18] and by A. Rahm and M. Fuchs [14, 15].)

The number-theoretic significance of the integral homology of arithmetic groups and their finite index subgroups is explained in the work of Ash, Gunnells and McConnell [3]. They require efficient techniques for computing (co)homology in order to investigate the conjecture that any Hecke eigenclass in the (co)homology of an arithmetic subgroup of $GL(m,\mathbb{Z})$ should have a Galois representation attached.

2. Perfect forms and a cell complex with $\mathrm{PSL}_4(\mathbb{Z})$-action.

In the space $S^n_{>0}$ of positive definite, real symmetric $n\times n$ matrices, we consider a specific $\mathrm{SL}_n(\mathbb{Z})$ invariant polyhedral complex. Such complexes are classically studied in the arithmetic theory of quadratic forms. One of them goes back to the work of Voronoi [21] on *perfect quadratic forms.*

For $A\in S^n_{>0}$ an associated positive definite quadratic form on $\mathbb{R}^n$ is defined by $A[x] = x^tAx$. By this correspondence we simply identify quadratic forms and symmetric matrices. The *arithmetical minimum* of $A\in S^n_{>0}$ is defined by

$$\min(A) = \min_{0\neq v\in\mathbb{Z}^n} A[v].$$

The finite set of its *representatives* is denoted by

$$\mathrm{Min}(A) = \{v\in\mathbb{Z}^n \text{ with } A[v]=\min(A)\}.$$

We consider the set of forms having a fixed arithmetical minimum, say 1:

$$S^n_{=1} = \{A \in S^n_{>0} \text{ with } \min(A) = 1\}.$$

This set is the boundary of a locally finite polyhedral set known as *Ryshkov polyhedron* (see [16] for details). In particular, $S^n_{=1}$ is a piecewise linear surface of dimension $\binom{n+1}{2} - 1$ and the support of a cell complex obtained from the faces of the Ryshkov polyhedron. Moreover, $S^n_{=1}$ is contractible. Each $v \in \mathbb{Z}^n$ determines one of the top-dimensional cells by the linear condition $A[v] = 1$. The 0-dimensional cells in $S^n_{=1}$ correspond to perfect quadratic forms. These are characterized as being uniquely determined by their arithmetical minimum (here 1) and its representatives.

The group $\mathrm{SL}_n(\mathbb{Z})$ acts on $S^n_{>0}$ and $S^n_{=1}$ by $A \mapsto PAP^t$. Some higher dimensional cells of $S^n_{=1}$ (and in particular all top-dimensional ones) have infinite stabilizer groups. To avoid problems arising from such stabilizers, we consider a deformation retract. A quadratic form $A \in S^n_{>0}$ is said to be *well-rounded* if there exist linearly independent $v_1, \ldots, v_n \in \mathrm{Min}(A)$. The set S^n_{wr} of well rounded forms in $S^n_{=1}$ defines an $\binom{n}{2}$-dimensional polyhedral subcomplex, in which all cells are bounded polyhedra and thus have finite stabilizer groups. By a result of Ash [1], S^n_{wr} is a homotopy retract of $S^n_{>0}$ (and $S^n_{=1}$) and is thus contractible. In fact, S^n_{wr} is a minimal deformation retract as there is no proper closed subset of S^n_{wr} which is $\mathrm{SL}_n(\mathbb{Z})$ invariant and contractible (see [13]).

The 0-dimensional cells of S^n_{wr} (the perfect forms) can be enumerated up to $\mathrm{SL}_n(\mathbb{Z})$ equivalence using *Voronoi's algorithm*. In essence it is a graph traversal search on the 1-dimensional subcomplex of $S^n_{=1}$ (see [21, 12, 16] for more details). Having a complete list of inequivalent perfect forms one can obtain all other orbits of cells, as these all contain at least one of the perfect forms.

In dimension 4 there are just two perfect forms up to $\mathrm{SL}_4(\mathbb{Z})$ equivalence (associated to the root lattices A_4 and D_4). These yield a $\mathrm{PSL}_4(\mathbb{Z})$-equivariant CW-decomposition of S^n_{wr} involving the following cells and cell stabilizer groups:

Dim. 0: two cell orbits with stabilizers A_5 and $(A_4 \times A_4) : C_2$.
Dim. 1: two cell orbits with stabilizers S_3 and $S_3 \times S_3$.
Dim. 2: two cell orbits, both with stabilizer $C_2 \times C_2$.
Dim. 3: four cell orbits, two with stabilizer $C_2 \times C_2$ and two with stabilizer S_4.
Dim. 4: four cell orbits with stabilizers S_3, D_8, S_4 and $C_2 \times S_3 \times S_3$.
Dim. 5: three cell orbits with stabilizers D_{24}, S_4 and A_5.
Dim. 6: one cell orbit with stabilizer $((C_2 \times C_2 \times C_2 \times C_2) : C_3) : C_2$.

Here we use $N : G$ to denote a semi-direct product of groups with G acting in N. The semi-direct product in dimension 0 is group [288,1026] in the GAP small groups library. The double semi-direct product in dimension 6 is group [96,227] in the library.

3. Resolutions for stabilizer subgroups

The method outlined in Section 4 for computing n terms of a free $\mathbb{Z}G$-resolution for $G = \mathrm{PSL}_4(\mathbb{Z})$ will require $n - p$ terms of a free $\mathbb{Z}G^e$-resolution for each stabilizer group of a p-dimensional cell in S^n_{wr}. An algorithm for computing a reasonably small free $\mathbb{Z}G^e$-resolution $R^{G^e}_* \to \mathbb{Z}$ for a generic finite group G^e was described in [8]. The implementation of this algorithm in [8] produces, for instance, a free $\mathbb{Z}A_5$-resolution with 26 free generators in dimension 6. The algorithm could be used to obtain $6-p$ terms for each of the stabilizer groups in the above tessellation of S^n_{wr}.

However, for computational efficiency it is necessary to craft smaller resolutions for some of the stabilizer groups G^e. One way to do this is to find a suitable polytope P on which G^e acts with small cell stabilizers. Then, using Lemma 4 below, we can combine resolutions for those subgroups of G^e stabilizing cells of P with the cellular chain complex $C_*(P)$ in order to produce a free $\mathbb{Z}G^e$-resolution $R^{G^e}_* \to \mathbb{Z}$. The idea was illustrated in [11] with the hand calculation of a free $\mathbb{Z}A_4$-resolution involving $n+1$ free generators in degree n. The same technique can be used, for instance, to construct a free $\mathbb{Z}A_5$-resolution involving $n+1$ free generators in degree n; in this case we take P to be the icosahedron and let A_5 act on P as a subgroup of the symmetry group $C_2 \times A_5$ of the icosahedron.

The first author used this polytopal method to craft small resolutions for the larger stabilizer groups in the tessellation of S^n_{wr}.

Both, the algorithm in [8] and the polytopal method [10, 11] can return a free $\mathbb{Z}G^e$-resolution $R^{G^e}_*$ endowed with a contracting homotopy. The latter is encoded as a sequence of $\mathbb{Z}$-linear homomorphisms $h_n\colon R^{G^e}_n \to R^{G^e}_{n+1}$ satisfying

$$h_{n-1}d_n + d_{n+1}h_n = 1 \quad (n \geq 0, h_{-1} = 0).$$

This contracting homotopy is required by Lemma 4 below.

4. Perturbation theory

It was observed in [10] that a method of Wall [22] for constructing free resolutions for group extensions can be extended to a method for constructing free resolutions for groups acting on contractible cellular spaces with nice stabilizer subgroups. We recall the method.

Suppose a chain complex

$$C_*\colon \cdots \to C_p \to C_{p-1} \to \cdots \to C_0 \to \mathbb{Z},$$

of $\mathbb{Z}G$-modules is given, where the modules C_p are not necessarily $\mathbb{Z}G$-free. Suppose also that for each $p \geq 0$ we are given a free $\mathbb{Z}G$-resolution $A_{p,*}$ of the module C_p

$$A_{p,*}\colon \to A_{p,q} \to A_{p,q-1} \to \cdots \to A_{p,0} \to C_p$$

endowed with a contracting homotopy $h_q\colon A_{p,q} \to A_{p,q+1}$ $(q \geq 0)$.

The following lemma explains how to construct a chain complex R_* of free $\mathbb{Z}G$-modules $R_n = \bigoplus_{p+q=n} A_{p,q}$ and a surjective $\mathbb{Z}G$-chain map

$$\phi_*\colon R_* \to C_*$$

which induces homology isomorphisms $H_n(\phi_*)\colon H_n(R_*) \xrightarrow{\cong} H_n(C_*)$ for $n \geq 0$. We state a slightly more general version of the lemma than was given in [10]. However, the proof is the same and boils down to the proof of the special case given in [22].

Lemma 4. [22, 10]

(i) *Let* $A_{p,q}$ $(p, q \geq 0)$ *be a bigraded family of free* $\mathbb{Z}G$*-modules. Suppose that there are* $\mathbb{Z}G$*-module homomorphisms* $d_0\colon A_{p,q} \to A_{p,q-1}$ *such that* $(A_{p,*}, d_0)$ *is an acyclic chain complex for each* p*. Set* $C_p = H_0(A_{p,*}, d_0)$ *and suppose further that there are* $\mathbb{Z}G$*-homomorphisms* $\delta\colon C_p \to C_{p-1}$ *for which* (C_*, δ) *is a chain complex. Then there exist* $\mathbb{Z}G$*-homomorphisms* $d_k\colon A_{p,q} \to A_{p-1,q+k-1}$ $(k \geq 1, p > k)$ *such that*

$$d = d_0 + d_1 + \cdots : R_n = \bigoplus_{p+q=n} A_{p,q} \to R_{n-1} = \bigoplus_{p+q=n-1} A_{p,q}$$

is a differential on a chain complex R_* *of free* $\mathbb{Z}G$*-modules.*

(ii) *The canonical chain maps* $\phi_p\colon A_{p,*} \to H_0(A_{p,*})$ *constitute a chain map* $\phi_*\colon R_* \to C_*$ *which is an homology isomorphism.*

(iii) *Suppose that there exist* $\mathbb{Z}$*-homomorphisms* $h_0\colon A_{p,q} \to A_{p,q+1}$ *such that* $d_0h_0d_0(x) = d_0(x)$ *for all* $x \in A_{p,q+1}$*. Then we can construct* d_k *by first lifting* δ *to* $d_1\colon A_{p,0} \to A_{p-1,0}$ *and recursively defining* $d_k = -h_0(\sum_{i=1}^{k} d_i d_{k-i})$ *on free generators of the module* $A_{p,q}$*.*

(iv) *Suppose that* C_* *is acyclic, that* $H_0(S_*) \cong \mathbb{Z}$ *and that each* C_p *is a free* $\mathbb{Z}$*-module. We can construct* $\mathbb{Z}$*-module homomorphisms* $h\colon R_n \to R_{n+1}$ *satisfying* $dhd(x) = d(x)$ *by setting* $h(a_{p,q}) = h_0(a_{p,q}) - hd^+h_0(a_{p,q}) + \epsilon(a_{p,q})$ *for free generators* $a_{p,q}$ *of the module* $A_{p,q}$*. Here* $d^+ = \sum_{i=1}^{p} d_i$ *and, for* $q \geq 1$*,* $\epsilon = 0$*. For* $q = 0$ *we define* $\epsilon = h_1 - h_0d^+h_1 + hd^+h_1 + hd^+h_0d^+h_1$ *where* $h_1\colon A_{p,0} \to A_{p+1,0}$ *is a* $\mathbb{Z}$*-linear homomorphism induced by a contracting homotopy on* C_**.*

For $G = \mathrm{PSL}_4(\mathbb{Z})$ and $X = S^n_{\mathrm{wr}}$ the cellular chain complex $C_*(X)$ can be viewed as a complex of $\mathbb{Z}G$-modules. Moreover,

$$C_p(X) \cong \bigoplus_{[e^p]} \mathbb{Z}G \otimes_{\mathbb{Z}G^{e^p}} \mathbb{Z}$$

where $[e^p]$ ranges over the orbits of p-dimensional cells. A free $\mathbb{Z}G$-resolution $A_{p,*}$ of $C_p(X)$ can thus be obtained as

$$A_{p,*} \cong \bigoplus_{[e^p]} R_*^{G^{e^p}} \otimes_{\mathbb{Z}G^{e^p}} \mathbb{Z}$$

where $R_*^{G^{e^p}}$ is any free $\mathbb{Z}G^{e^p}$-resolution of $\mathbb{Z}$. Moreover, contracting homotopies on $R_*^{G^{e^p}}$ induce a contracting homotopy on $A_{p,*}$. Lemma 4 thus gives an automated procedure to combine the chain complex $C_*(X)$ with resolutions for the stabilizer groups to produce a free $\mathbb{Z}G$-resolution R_*^G of $\mathbb{Z}$.

The first six terms of such a resolution R_*^G were computed and, via the Smith Normal Form algorithm, used to prove Theorem 1(i).

5. The Leray spectral sequence

Details on the Leray spectral sequence can be found in [4] (pp. 173-174). For $G = \mathrm{PSL}_4(\mathbb{Z})$ and $X = S^n_{\mathrm{wr}}$ it has the form

$$E^1_{p,q} = \bigoplus_{[e^p]} H_q(G^{e^p}, \mathbb{Z}^\rho) \Rightarrow H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})$$

where $[e^p]$ ranges over the G-orbits of p-dimensional cells e^p in the tessellation of X and the integer coefficients are twisted by an action $\rho\colon G^{e^p} \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$.

Each stabilizer group is finite. The homology of any finite group is finite in degrees ≥ 1. So since X is 6-dimensional the spectral sequence immediately implies that $H_n(G, \mathbb{Z})$ is finite for $n > 6$. A computer calculation shows that $E^2_{6,0}$ is finite. Therefore $H_6(G, \mathbb{Z})$ is finite and Theorem 1(iv) follows from Theorem 1(i).

No prime $p \geq 7$ divides the order of any of the stabilizer subgroups G^e and thus no $p \geq 7$ divides the order of $H_q(G^e, \mathbb{Z}^\rho)$. So Theorem 1(iii) follows from the spectral sequence.

The only 3-groups and 5-groups arising as subgroups of stabilizer groups are C_3, $C_3 \times C_3$ and C_5. These prime-power groups can only act trivially on $\mathbb{Z}$. Using Cartan and Eilenberg's identification of the p-part $H^q(G^e, \mathbb{Z}^\rho)_{(p)}$ of the cohomology of G^e with the G-stable elements in the cohomology $H^q(\mathrm{Syl}_p(G^e), \mathbb{Z})$ of the Sylow p-subgroup, it is straightforward to determine $H_{q-1}(G^e, \mathbb{Z}^\rho)_{(p)} = H^q(G^e, \mathbb{Z}^\rho)_{(p)}$ for $p = 3, 5$. (The action ρ is important here. For instance, in degree 3 one of the S_4 stabilizer groups acts trivially on $\mathbb{Z}$; for the other S_4 stabilizer group all odd permutations acts non-trivially on $\mathbb{Z}$. This results in the strange looking isomorphism $H_q(S_4, \mathbb{Z}^\rho)_{(3)} \oplus H_q(S_4, \mathbb{Z}^\rho)_{(3)} \cong \mathbb{Z}_3$ for odd q.) The 3-part is presented in Proposition 2. The 5-part is given in the following.

Proposition 5. *At the prime 5 the Leray spectral sequence $E^1_{p,q} \Rightarrow H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)}$ has first page*

$$E^1_{p,q} = \begin{cases} \mathbb{Z}_5 & p = 0, q = 3 + 4k \\ \mathbb{Z}_5 & p = 5, q = 3 + 4k \\ 0 & \text{otherwise} \end{cases} \qquad (k \geq 0).$$

Proposition 5 directly implies that

$$H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)} = \begin{cases} 0 \text{ or } \mathbb{Z}_5 & \text{if } n \equiv 0, 3 \bmod 4 \ (n \geq 6) \\ 0 & \text{if } n \equiv 1, 2 \bmod 4 \ (n \geq 6) \end{cases}$$

The alternating group A_5 is the stabilizer group of a 0-cell of X. A computer calculation shows that the inclusion $A_5 \hookrightarrow \mathrm{PSL}_4(\mathbb{Z})$ induces a surjection in cohomology $H^4(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)} \to H^4(A_5, \mathbb{Z})_{(5)}$. The ring $H^*(A_5, \mathbb{Z})_{(5)} \cong \mathbb{Z}_5[x]$ is generated by a single class x in degree 4. Hence the ring homomorphism $H^*(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)} \to H^*(A_5, \mathbb{Z})_{(5)}$ is surjective and consequently $H_n(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)} \cong H^{n+1}(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)}$ is non-trivial for $n \equiv 3 \bmod 4$ ($n \geq 7$). This proves Theorem 1(ii).

We remark that in degree 4 the 5-part $H_4(\mathrm{PSL}_4(\mathbb{Z}), \mathbb{Z})_{(5)} = \mathbb{Z}_5$ arises from the torsion free term $E_{4,0}$.

6. The groups $\mathrm{PGL}_3(\mathbb{Z}[i])$ and $\mathrm{PGL}_3(\mathbb{Z}[\omega])$

The above techniques apply to other arithmetic groups. In particular, the first author using the construction of [17] for equivariant perfect forms has computed two 8-dimensional contractible CW-spaces on which the groups $\mathrm{PGL}_3(\mathbb{Z}[i])$ and $\mathrm{PGL}_3(\mathbb{Z}[\omega])$ act cellularly with finite stabilizer groups. These spaces, together with the above 6-dimensional space for $\mathrm{PSL}_4(\mathbb{Z})$, have been stored in the HAP package [9] for the GAP computer algebra system.

The following short GAP session illustrates how the space for $\mathrm{PGL}_3(\mathbb{Z}[i])$ can be accessed and used to compute: (i) five terms of a free resolution and (ii) the fourth integral homology group.

```
gap> C:=ContractibleGcomplex("PGL(3,Z[i])");;
gap> R:=FreeGResolution(C,5);;
gap> Homology(TensorWithIntegers(R),4);
[ 2, 0 ]
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The resolution can also be used to determine the homology of finite index subgroups of $\mathrm{PGL}_3(\mathbb{Z}[i])$, though GAP's standard implementation of the Smith Normal Form algorithm does not work well when the index is large.

For the group $\mathrm{Sp}_4(\mathbb{Z})$ the 4-dimensional CW-complex of [20], which comes also from the perfect forms in dimension 4, has been stored in the HAP package [9]. So, for instance, the following GAP session computes the homology $H_3(\mathrm{Sp}_4(\mathbb{Z}),\mathbb{Z}) = (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{240}$ in agreement with calculations of [5].

```
gap> C:=ContractibleGcomplex("Sp(4,Z)");;
gap> R:=FreeGResolution(C,4);;
gap> Homology(TensorWithIntegers(R),3);
[ 2, 2, 12, 240 ]
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