

WYTHOFF POLYTOPES AND LOW-DIMENSIONAL HOMOLOGY OF MATHIEU GROUPS

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ABSTRACT. We describe two methods for computing the low-dimensional integral homology of the Mathieu simple groups and use them to make computations such as $H_5(M_{23}, \mathbb{Z}) = \mathbb{Z}_7$ and $H_3(M_{24}, \mathbb{Z}) = \mathbb{Z}_{12}$. One method works via Sylow subgroups. The other method uses a Wythoff polytope and perturbation techniques to produce an explicit free $\mathbb{Z}M_n$ -resolution. Both methods apply in principle to arbitrary finite groups.

1. INTRODUCTION

We describe two methods for computing the integral homology for the Mathieu simple groups presented on Table 1. The first homology $H_1(G, \mathbb{Z})$ is trivial for any simple group and so is omitted from the table (see [3] for an exposition of relevant facts on group homology). The second homology of Mathieu groups is well-known [16]. A computer method for the second homology of a permutation group was illustrated on the Mathieu groups M_{21} and M_{22} in [15]. The mod p cohomology $H^*(G, \mathbb{F}_p)$ is now known for all Mathieu groups except M_{24} [21, 1, 2, 17]. With the help of the Bockstein spectral sequence it is, in principle, possible to obtain integral homology from mod p cohomology (p ranging over the prime divisors of the group order), though the details can be difficult. For example, the calculation of $H_n(M_{23}, \mathbb{Z})$ was obtained in this way for $1 \leq n \leq 6$ by Milgram [17] and provided the first example of a non-trivial finite group with trivial integral homology in dimensions ≤ 3 . It seems that the mod p cohomology of M_{24} is not known for all primes p (see [14] for the case $p = 3$) and so we can assign the status of a new theorem to the following result.

Theorem 1. $H_3(M_{24}, \mathbb{Z}) = \mathbb{Z}_{12}$ and $H_4(M_{24}, \mathbb{Z}) = 0$.

This result (and other table entries) can be obtained from the HAP homological algebra package [10] for the GAP computational algebra system [12] using (variants of) the following command.

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gap> GroupHomology(MathieuGroup(24), 3);  
gap> [ 4, 3 ]
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The algorithm underlying this command is explained in Section 2. The current implementation is unable to determine the integers a, b, c in Table 1 though it does establish the ranges $0 \leq a \leq 53$, $0 \leq b \leq 1$.

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G	$H_2(G, \mathbb{Z})$	$H_3(G, \mathbb{Z})$	$H_4(G, \mathbb{Z})$	$H_5(G, \mathbb{Z})$
M_{11}	0	\mathbb{Z}_8	0	\mathbb{Z}_2
M_{12}	\mathbb{Z}_2	$\mathbb{Z}_6 \oplus \mathbb{Z}_8$	\mathbb{Z}_3	$(\mathbb{Z}_2)^3$
M_{21}	$\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	\mathbb{Z}_5	0	$(\mathbb{Z}_2)^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7$
M_{22}	\mathbb{Z}_{12}	0	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_7$
M_{23}	0	0	0	\mathbb{Z}_7
M_{24}	0	\mathbb{Z}_{12}	0	$(\mathbb{Z}_2)^a \oplus (\mathbb{Z}_4)^b \oplus \mathbb{Z}_7$

TABLE 1. Low dimensional homology of Mathieu groups with $0 \leq a \leq 53$ and $0 \leq b \leq 1$.

Abelian invariants of a (co)homology group are the easiest cohomological information to access. More difficult information would be, for example, explicit cocycles $G^n \rightarrow A$ corresponding to cohomology classes in $H^n(G, A)$. Explicit cocycles are constructed in HAP using the induced chain map $B_*^G \rightarrow R_*^G$ from the bar resolution B_*^G to an explicit small free $\mathbb{Z}G$ -resolution R_*^G of \mathbb{Z} . In Sections 3-5 we explain how the Wythoff polytope construction can be used to produce such a resolution R_*^G . This resolution provides an alternative proof of Theorem 1.

In Section 6 we determine the p -part $H_n(M_m, \mathbb{Z})_{(p)}$ of the integral homology of the Mathieu groups for $n \geq 1$ and primes $p \geq 5$. For $p \in \{5, 7, 11, 23\}$ the p -part is either trivial or \mathbb{Z}_p ; it is trivial for all other primes $p \geq 5$. Table 2 lists the values of n for which the p -part is non-trivial.

Although the paper focuses on Mathieu groups, the techniques are applicable in principle to arbitrary finite groups. In some cases the Wythoff polytopal method is a significantly faster method for computing the homology groups.

2. ALGORITHM UNDERLYING THE HAP FUNCTION

Given a group G , a *free $\mathbb{Z}G$ -resolution* of the trivial module \mathbb{Z} is an exact sequence

$$0 \leftarrow \mathbb{Z} \leftarrow R_0^G \leftarrow R_1^G \leftarrow \dots \leftarrow R_k^G \leftarrow \dots$$

of free $\mathbb{Z}G$ -modules R_i^G . A previous paper [9] describes an algorithm for computing free $\mathbb{Z}G$ -resolutions for finite G . This has now been implemented as part of the HAP package. It takes as input a finite group G and a positive integer n . It returns:

- The rank of the k th module R_k^G in a free $\mathbb{Z}G$ -resolution R_*^G ($0 \leq k \leq n$).
- The image of the i th free $\mathbb{Z}G$ -generator of R_k^G under the boundary homomorphism $d_k: R_k^G \rightarrow R_{k-1}^G$ ($1 \leq k \leq n$).
- The image of the i th free \mathbb{Z} -generator of R_k^G under a contracting homotopy $h_k: R_k^G \rightarrow R_{k+1}^G$ ($0 \leq k \leq n-1$).

The contracting homotopy satisfies, by definition, $h_k d_{k+1} + d_{k+2} h_{k+1} = 1$ and needs to be specified on a set of free Abelian group generators of R_k since it is not G -equivariant. The

$H_n(M_m, \mathbb{Z})_{(p)} = \mathbb{Z}_p$	$m = 11$	$m = 12$	$m = 21$	$m = 22$	$m = 23$	$m = 24$
	$n =$	$n =$	$n =$	$n =$	$n =$	$n =$
$p = 5$	$8k - 1$	$8k - 1$	$4k - 1$	$8k - 1$	$8k - 1$	$8k - 1$
$p = 7$	—	—	$6k - 1$	$6k - 1$	$6k - 1$	$6k - 1$
$p = 11$	$10k - 1$	$10k - 1$	—	$10k - 1$	$10k - 1$	$20k - 1$
$p = 23$	—	—	—	—	$22k - 1$	$22k - 1$

 TABLE 2. Values of n expressed in term of $k \geq 1$ such that $H_n(M_m, \mathbb{Z})_{(p)} = \mathbb{Z}_p$

homotopy can be used to make constructive the following frequent element of choice.

For $x \in \ker(d_k: R_k^G \rightarrow R_{k-1}^G)$ choose an element $\tilde{x} \in R_{k+1}^G$ such that $d_{k+1}(\tilde{x}) = x$.

One sets $\tilde{x} = h_k(x)$. In particular, for any group homomorphism $\phi: G \rightarrow G'$, the homotopy allows one to define an induced ϕ -equivariant chain map $\phi_*: R_*^G \rightarrow R_*^{G'}$.

The algorithm in [9] can only handle fairly small groups. For example, the HAP implementation takes 20 seconds on a 2.66GHz Intel PC with 2G of memory to compute eight terms of a free $\mathbb{Z}G$ -resolution R_*^G for the symmetric group $G = S_5$; the $\mathbb{Z}G$ -rank of R_8^G is 115. However, for any group G there is a surjection

$$H_n(\text{Syl}_p, \mathbb{Z}) \rightarrow H_n(G, \mathbb{Z})_{(p)}$$

from the homology of a Sylow p -subgroup $\text{Syl}_p = \text{Syl}_p(G)$ onto the p -part of the homology of G . For a Sylow p -subgroup P there is a description of the kernel of the surjection $H_n(P, \mathbb{Z}) \rightarrow H_n(G, \mathbb{Z})_{(p)}$ due to Cartan and Eilenberg [4]. It is generated by elements

$$\phi_K(a) - \phi_{xKx^{-1}}(a)$$

where x ranges over the double coset representatives of P in G , $K = P \cap xPx^{-1}$, the homomorphisms $\phi_K, \phi_{x^{-1}Kx}: H_n(K, \mathbb{Z}) \rightarrow H_n(P, \mathbb{Z})$ are induced by the inclusion $K \rightarrow P, k \mapsto k$ and the conjugated inclusion $K \rightarrow P, k \mapsto x^{-1}kx$, and a ranges over the generators of $H_n(K, \mathbb{Z})$. Thus, the homology of a large finite group G can be computed from free resolutions (with specified contracting homotopy) for each of its Sylow subgroups. Our implementation of the algorithm in [9] can be used to produce six terms of free $\mathbb{Z}(\text{Syl}_p)$ -resolutions for all Sylow subgroups Syl_p of all Mathieu groups except M_{24} . The Sylow subgroup $\text{Syl}_2(M_{24})$ has order 1024 and requires a specific application of a general technique.

To explain the technique suppose that G is a group, possibly infinite, for which we have some $\mathbb{Z}G$ -resolution of \mathbb{Z}

$$C_*: \cdots \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_0 \rightarrow \mathbb{Z}.$$

but that C_* is not free. Suppose that for each m we have a free $\mathbb{Z}G$ -resolution of the module C_m

$$D_{m*}: \cdots \rightarrow D_{m,n} \rightarrow D_{m,n-1} \rightarrow \cdots \rightarrow D_{m,0} \rightarrow C_m.$$

p	$H_1(\text{Syl}_p, \mathbb{Z})$	$H_2(\text{Syl}_p, \mathbb{Z})$	$H_3(\text{Syl}_p, \mathbb{Z})$	$H_4(\text{Syl}_p, \mathbb{Z})$	$H_5(\text{Syl}_p, \mathbb{Z})$
2	$(\mathbb{Z}_2)^4$	$(\mathbb{Z}_2)^8$	$(\mathbb{Z}_2)^{11} \oplus (\mathbb{Z}_4)^6$	$(\mathbb{Z}_2)^{32}$	$(\mathbb{Z}_2)^{52} \oplus \mathbb{Z}_4$
3	$(\mathbb{Z}_3)^2$	$(\mathbb{Z}_3)^2$	$(\mathbb{Z}_3)^4$	$(\mathbb{Z}_3)^3$	$(\mathbb{Z}_3)^4 \oplus \mathbb{Z}_9$

TABLE 3. Low dimensional homology of Sylow subgroups of M_{24} for $p = 2, 3$

Theorem 2. [20] *There is a free $\mathbb{Z}G$ -resolution $R_*^G \rightarrow \mathbb{Z}$ with*

$$R_n^G = \bigoplus_{p+q=n} D_{p,q}$$

The proof of this theorem of C.T.C. Wall can be made constructive by using contracting homotopies on the resolutions D_{m*} . Furthermore, a contracting homotopy on R_*^G can be constructed by a formula involving contracting homotopies on the D_{m*} and on C_* . Details are given in [11].

Suppose now that N is a normal subgroup of G and that C_* is a free $\mathbb{Z}(G/N)$ -resolution. Then, regarding C_* as a $\mathbb{Z}G$ -resolution, each free $\mathbb{Z}G$ -generator of C_m is stabilized by N . Any free $\mathbb{Z}N$ -resolution of \mathbb{Z} can be used to construct a free $\mathbb{Z}G$ -resolution D_{m*} of C_m . Thus, using Theorem 2, we can construct a free $\mathbb{Z}G$ -resolution R_*^G from a free $\mathbb{Z}N$ -resolution R_*^N and free $\mathbb{Z}(G/N)$ -resolution $R_*^{G/N}$. The constructed resolution is often referred to as a *twisted tensor product* and denoted by $R_*^G = R_*^N \tilde{\otimes} R_*^{G/N}$.

This twisted tensor product has been implemented in HAP and can be used to provide free resolutions for the Sylow subgroup $\text{Syl}_p(M_{24})$. Since $|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ the non-cyclic Sylow subgroups occur only for $p = 2, 3$. Their low-dimensional integral homology can be computed using HAP and is given in Table 3.

In degrees $n = 5$ the current version of HAP fails to determine the image of $H_n(\text{Syl}_2, \mathbb{Z})$ in $H_n(M_{24}, \mathbb{Z})$. It succeeds in constructing the image as a finitely presented group but fails to determine the group from this presentation. This failure should be resolved in a future release of HAP.

The remainder of the paper is aimed at constructing small free resolutions for large groups such as M_{24} .

3. ORBIT POLYTOPES

Suppose that a finite group G acts linearly on \mathbb{R}^n . For a vector $v \in \mathbb{R}^n$ we consider the convex hull

$$P = P(G, v) = \text{Conv}(v^g : g \in G)$$

of the orbit of v under the action of G . The polytope P has a natural cell structure with respect to which we can consider the cellular chain complex $C_*(P)$. The action of G on \mathbb{R}^n induces an action of G on $C_*(P)$ and we can view $C_*(P)$ as a chain complex of $\mathbb{Z}G$ -modules. Since P is contractible we have $H_i(C_*(P)) = 0$ for all $i \geq 1$ and $H_0(C_*(P)) = \mathbb{Z}$. Furthermore, if the polytope is of dimension m then $H_0(C_*(P)) \cong \mathbb{Z} \cong (C_m(P))$. So there

is a homomorphism $C_0(P) \rightarrow C_{m-1}(P)$ which can be used to splice together infinitely many copies of $C_*(P)$ to form an infinite $\mathbb{Z}G$ -resolution

$$\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow C_{m-1} \rightarrow \cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow \mathbb{Z}$$

of the trivial $\mathbb{Z}G$ -module \mathbb{Z} . In principle one can use Theorem 2 to convert C_* to a free $\mathbb{Z}G$ -resolution. Precise details are given in [11]. To put this idea into practice one requires:

- (1) The face lattice of the orbit polytope $P(G, v)$.
- (2) For each orbit of cell e in $P(G, v)$, the subgroup $\text{Stab}(G, e) \leq G$ of elements that stabilize e globally.
- (3) A free $\mathbb{Z}\text{Stab}(G, e)$ -resolution $R_*^{\text{Stab}(G, e)}$ for each stabilizer $\text{Stab}(G, e)$.

Assuming that the stabilizer groups $\text{Stab}(G, e)$ are reasonably small, resolutions $R_*^{\text{Stab}(G, e)}$ are readily obtained from HAP's implementation of the algorithm in [9]. Thus, to convert C_* to a free $\mathbb{Z}G$ -resolution, we must focus on requirements (1) and (2).

One could use computational geometry software such as Polymake [13] to determine the combinatorial structure of $P(G, v)$ for small groups G . For instance, any permutation group $G \leq S_n$ acts on \mathbb{R}^n by $\pi(x_1, \dots, x_n) = (x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(n)})$ for $\pi \in G$. In particular, the Mathieu group M_{10} of order 720, generated by $\pi_1 = (1, 9, 6, 7, 5)(2, 10, 3, 8, 4)$ and $\pi_2 = (1, 10, 7, 8)(2, 9, 4, 6)$, acts on \mathbb{R}^{10} . For the vector $v = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$ the polytope $P(M_{10}, v)$ is 9-dimensional with 720 vertices each of degree 632. The polytope thus has 227520 edges.

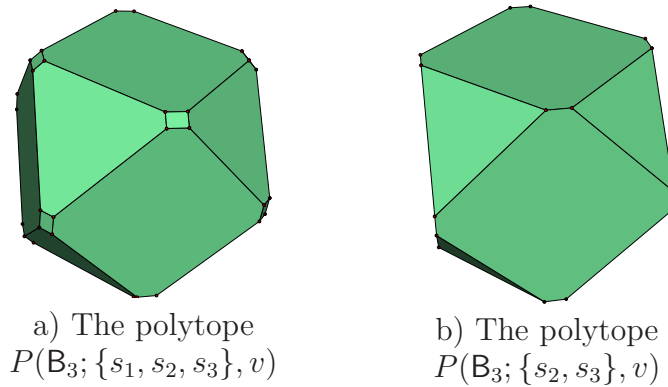
4. ORBIT POLYTOPES OF FINITE REFLECTION GROUPS

Let W be a finite reflection group generated by a simple system of Euclidean reflections $S = \{s_1, \dots, s_n\}$. For each reflection $s \in S$ let H_s denote the corresponding reflecting hyperplane and Δ the fundamental simplex for S . The Coxeter-Dynkin graph is the graph on S with two reflections adjacent if they do not commute. Fix a subset $\emptyset \subsetneq V \subseteq S$. The *type* $T = t(v) \subset S$ of a point $v \in \Delta$ is the set of $s \in S$ such that $v \notin H_s$. Choose a point v of type V . Let $P(W; V, v)$ denote the n -dimensional polytope formed by the convex hull of the orbit of v under the action of W .

As an example, consider the 3-dimensional reflection group $W = B_3$ generated by reflections s_1, s_2, s_3 where $(s_1 s_2)^3 = 1$, $(s_1 s_3)^2 = 1$ and $(s_2 s_3)^4 = 1$. For $V = \{s_1, s_2, s_3\}$ and vector $v \in \mathbb{R}^3$ in general position but close to the mirrors H_{s_1} and H_{s_3} the polytope $P(W; V, v)$ is pictured in Figure 1.a). For $V = \{s_2, s_3\}$ and $v \in H_{s_1}$ the polytope $P(W; V, v)$ is pictured in Figure 1.b).

Proposition 3. *The combinatorial type of $P(W; V, v)$ is independent of the choice of v .*

Proof. For $V = S$ the polytope obtained is the well-known permutahedron, whose face-lattice is independent of v . The stabilizer of a face of $P(W; V, v)$ is a parabolic subgroup and this establishes an isomorphism between the face lattice and the lattice of parabolic subgroups. Furthermore, the 1-skeleton of $P(W; S, v)$ is the Cayley graph of W with respect to the generating set S . Observe that in this Cayley graph the length of any edge

FIGURE 1. Two Wythoff polytopes constructed from D_3

labelled by generator $s \in S$ decreases as v is moved towards the hyperplane H_s , and that the edge ceases to exist when v moves into H_s . The set of vectors v of type $t(v) = S$ has measure one in the set of all vectors. Thus any face F of $P(W; V, v)$ is obtained as the limit of some face G of the permutahedron and the set of vertices contained in F is the limit of the vertices contained in G . Thus the Cayley graph determines the face-lattice and this proves the required result. \square

We are going to describe explicitly the face lattice $P_{W,V}$ of the polytope $P(W; V, v)$ for any v of type V . Our description is of course equivalent to the classic one given in [5, 6, 22] and reproduces the one of [19]. Since we assume that W is finite, the Coxeter-Dynkin graph of W is a tree and given any two vertices u and v of it we denote by $[u, v]$ the unique path from u to v .

For two subsets $U, U' \in S$ we say that U' *blocks* U (from V) if for all $u \in U$ and $v \in V$ there is a $u' \in U'$, such that $u' \in [u, v]$. This defines a binary relation on subsets of S , which we will denote as $U' \leq U$. We also write $U' \sim U$ if $U' \leq U$ and $U \leq U'$, and we write $U' < U$ if $U' \leq U$ and $U \not\leq U'$.

It is easy to see that \leq is reflexive and transitive, which implies that \sim is an equivalence relation. Let $[U]$ denote the equivalence class containing U . It can be shown that if $U \sim U'$ then $U \cap U' \sim U \sim U \cup U'$. This yields that every equivalence class X contains a unique smallest (under inclusion) subset $m(X)$ and unique largest subset $M(X)$. The subsets $m(X)$ will be called the *essential* subsets of S (with respect to V). Let $E(V)$ be the set of all essential subsets of S . Clearly, the above relation $<$ is a partial order on $E(V)$. Also, $V \in E(V)$ and V is the smallest element of $E(V)$ with respect to $<$.

The faces F of $P(W; V, v)$ are indexed by their isobarycenters $g(F)$. The stabilizer of F is the stabilizer of $g(F)$, that is the parabolic subgroup of W generated by $S - t(g(F))$. The type of such an isobarycenter is an essential subset of S and all essential subsets are realized as isobarycenters of faces. The rank of an essential subset is the dimension of the corresponding face. Given two faces F, F' of $P(W; V, v)$, $F \subset F'$ if and only if we have the type inequality $t(g(F)) < t(g(F'))$ and $\{g(F), g(F')\}$ is contained in at least one image $g(\Delta)$ with $g \in W$ of the fundamental simplex Δ .

We can use the above formalism to obtain the combinatorial structure of the orbit polytope $P(M_{24}, v)$ where the Mathieu group acts on \mathbb{R}^{24} by permuting basis vectors, and $v = (1, 2, 3, 4, 5, 0, \dots, 0) \in \mathbb{R}^{24}$. Since M_{24} is a 5-transitive permutation group we have

$$P(M_{24}, v) = P(S_{24}, v).$$

The symmetric group S_{24} is a finite reflection group with simple generating system $S = \{s_i = (i, i + 1) : 1 \leq i \leq 23\}$. The vector v lies in those mirrors H_{s_i} for $6 \leq i \leq 23$. So $P(M_{24}, v) = P(S_{24}, V, v)$ for $V = \{s_1, \dots, s_5\}$.

Our proof of proposition 3 implies that the polytope $P(M_{24}, v)$ has $|S_{24}/\langle s_i : 6 \leq i \leq 23 \rangle| = 5100480$ vertices. The essential subsets of rank 1 defining edges are $V - \{s_k\}$ for $1 \leq k \leq 4$ and $(V - \{s_5\}) \cup \{s_6\}$. So, the number of edges is

$$|S_{24}/\langle s_5, s_i : 7 \leq i \leq 23 \rangle| + \sum_{1 \leq k \leq 4} |S_{24}/\langle s_k, s_i : 6 \leq i \leq 23 \rangle| = 58655520.$$

Each vertex of the polytope has the same degree d say. Thus the number of edges is $d \times 5100480/2 = 58655520$ from which $d = 23$. Since $P(M_{24}, v)$ is of dimension 23, this shows that it is simple.

Each vertex of $P(M_{24}, v)$ has stabilizer group $\text{Stab}(M_{24}, v) = M_{24} \cap \langle s_i : 6 \leq i \leq 23 \rangle \cong (C_2 \times C_2 \times C_2 \times C_2) : C_3$ of order 48. Under M_{24} , for $1 \leq k \leq 4$, there is only one orbit of edges of type $V - \{s_k\}$; they have stabilizer $\text{Stab}(M_{24}, v) : C_2$ of order 96. Under M_{24} there are two orbits of edges of type $(V - \{s_5\}) \cup \{s_6\}$, one with stabilizer S_3 , the other with stabilizer a 2-group of order 32.

The formalism of essential subsets is a useful tool to determine the face lattice of $P(W; V, v)$ for a Coxeter group W and provides ready access to the lattice for cohomology calculations. The equality between the polytopes $P(M_{24}, v)$ and $P(S_{24}, v)$ was essential for getting a reasonably simple description of the face lattice.

For an arbitrary vector v and group G we cannot expect to have a simple combinatorial description of the face lattice and we need to use specific computational techniques. If G is large, we cannot expect to be able to store the vertex set of $P(G, v)$. Fortunately, by the group action, the full face lattice is encoded in the set $S(v)$ of vertices adjacent to v . This set $S(v)$ can be computed iteratively by using the Poincaré polyhedron theorem [18, 7]. Once the list of neighbours is known the face-lattice follows easily.

After one has obtained the low dimensional faces of $P(M_{24}, v)$ and their stabilizer groups, we can use Theorem 2 to compute the initial terms of a free $\mathbb{Z}M_{24}$ -resolution of \mathbb{Z} .

5. WYTHOFF CONSTRUCTION FOR POLYTOPES

The Wythoff construction can also be defined for partially ordered sets. A *flag* in a poset is an arbitrary completely ordered subset. We say that a connected poset \mathcal{K} is a *d-dimensional complex* (or, simply, a *d-complex*) if every maximal flag in \mathcal{K} has size $d + 1$. In a *d-complex* \mathcal{K} every element x can be uniquely assigned a number $\dim(x) \in \{0, \dots, d\}$, called the *dimension* of x , in such a way, that the minimal elements of \mathcal{K} have dimension zero and $\dim(y) = \dim(x) + 1$ whenever $x < y$ and there is no z with $x < z < y$. The elements of a complex \mathcal{K} are called *faces*, or *k-faces* if the dimension of the face needs to be

G	P	V	Free rank of resolution in degrees $0, 1, 2, \dots$
M_{22}	α_{21}	$\{0, 1, 2\}$	1, 7, 33, 113, 301, 694
M_{23}	α_{22}	$\{0, 1, 2, 3, 4\}$	2, 20, 116, 451, 1334, 3279
M_{24}	α_{23}	$\{0, 1, 2, 3, 4\}$	1, 9, 50, 204, 649

TABLE 4. Rank of resolutions of M_{22} , M_{23} , M_{24} obtained from the Wythoff construction

specified. Furthermore, 0-faces are called *vertices* and d -faces (maximal faces) are called *facets*. If $x < y$ and $\dim(x) = k$, we will say that x is a k -face of y .

For a flag $F \subset \mathcal{K}$ define its *type* as the set $t(F) = \{\dim(x) : x \in F\}$. Clearly, $t(F)$ is a subset of $S = \{0, \dots, d\}$ and, conversely, every subset of S is the type of some flag. Let Ω be the set of all nonempty subsets of S and fix an arbitrary $V \in \Omega$. For two subsets $U, U' \in \Omega$ we say that U' *blocks* U (from V) if for all $u \in U$ and $v \in V$ there is a $u' \in U'$ and $u \leq u' \leq v$ or $v \leq u' \leq u$. With this notion of blocking we can define the notion of essential subset of S and the inequality $<$ in the same way as for Coxeter groups.

The construction of $P(\mathcal{K}; V)$ mimics the one of $P(W; D, v)$ above for Coxeter groups. The *Wythoff complex* $P(\mathcal{K}; V)$ consists of all flags F such that $t(F)$ is essential. For two such flags F and F' , we have $F' < F$ whenever $t(F') < t(F)$ and F' is compatible with F , that is, $F \cup F'$ is a flag. It can be shown that $P(\mathcal{K}, V)$ is again a d -complex.

The face lattice $\mathcal{K}(P)$ of a $(d+1)$ -dimensional polytope P is a d -complex, which is a CW-complex topologically equivalent to a sphere. It is proved in [19] that the topological type of $P(\mathcal{K}; V)$ is the same as the one of \mathcal{K} . This version of the Wythoff construction when applied to a regular polytope gives a face lattice which is isomorphic to the one obtained by applying the Wythoff construction to the corresponding Coxeter group. The complex $P(\mathcal{K}(P), \{0\})$ is equal to $\mathcal{K}(P)$ and $P(\mathcal{K}(P), \{d\})$ is the complex of the polytope dual to P . In general $P(\mathcal{K}(P), V)$ is not a polytope since the notion of convexity is not well preserved by the Wythoff construction without any regularity assumption.

The topological invariance means that if a group G acts on a polytope P then we can apply the orbit polytope construction to $P(\mathcal{K}(P), V)$ for a chosen V in order to compute $H_i(G, \mathbb{Z})$.

In the case of M_{24} , we take as polytope the 23-dimensional simplex α_{23} and we build the Wythoff polytope $P(\alpha_{23}; \{1, 2, 3, 4, 5\})$. The method applies to any finite group acting on n points by using the simplex α_{n-1} . We do not need G to act transitively.

In Table 4 we give the results obtained for the larger Mathieu groups. All programs are available from [8].

6. HOMOLOGY AT $p = 5, 7, 11, 23$

Suppose that a group G has Sylow p -subgroup $P = C_p$ of prime order. The Cartan-Eilenberg double coset formula implies that the surjection

$$\pi_n: H_n(P, \mathbb{Z}) \rightarrow H_n(G, \mathbb{Z})_{(p)}$$

has kernel generated by the elements

$$H_n(\phi_g)(a) - a$$

for $g \in N_G(P)$, $a \in H_n(P, \mathbb{Z})$ and $\phi_g: P \rightarrow P, p \mapsto gpg^{-1}$. Here $N_G(P)$ is the normalizer of P in G .

Using the isomorphism $H_{n-1}(P, \mathbb{Z}) \cong H^n(P, \mathbb{Z})$ and the cohomology ring structure $H^*(P, \mathbb{Z}) \cong \mathbb{Z}_p[x^2]$, we see that a group homomorphism $\phi: P \rightarrow P, p \mapsto p^m$ induces a homology homomorphism $H_{2k-1}(\phi): H_{2k-1}(P, \mathbb{Z}) \rightarrow H_{2k-1}(P, \mathbb{Z}), a \mapsto a^{m^k}$.

For $p \in \{5, 7, 11, 23\}$ the Mathieu groups have Sylow p -subgroups which are either trivial or of prime order. One can use GAP to determine their normalizers. It is thus a routine exercise to determine the p -part of the integral homology of the Mathieu groups, the results of which are given in the Introduction.

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