Homology of some Artin and twisted Artin Groups

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Abstract

We begin the paper with a simple formula for the second integral homology of a range of Artin groups. The formula is derived from a polytopal classifying space. We then introduce the notion of a *twisted Artin group* and obtain polytopal classifying spaces for a range of such groups. We demonstrate that these explicitly constructed spaces can be implemented on a computer and used in homological calculations.

Mathematics Subject Classification: 20J06

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1 Introduction

Any relation between generators of a group can be drawn as a polygon. A "relation among relations", or "non-commutative syzygy among the relations", can be viewed as a 3-dimensional polytope with boundary composed of such polygons. Higher-order non-commutative syzygies of a group can be treated as higher-dimensional polytopes. A formal account of these ideas is given in the paper [20] by M. Kapranov and M. Saito. That paper focuses on groups of elementary matrices over a ring and asserts "that the study of higher syzygies among row operations is, at least, ideally, the aim of algebraic K-theory". The present paper focuses on higher syzygies in a different class of groups containing the braid groups, the Baumslag-Solitar groups and many Bieberbach groups.

An *Artin group* is a finitely presented group with one relator

$$(xy)_m = (yx)_m$$

between each pair of distinct generators x, y. Here m depends on x and y and $(xy)_m = xyx...$ denotes the word of length m or, when $m = \infty$, denotes the trivial word. The starting point for this paper is a surprisingly simple formula (Theorem 1)

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for the second integral homology of a range of Artin groups. Although the formula is apparently new, its proof is essentially an observation about the polytopal structure of the 3-dimensional cells of a much studied CW-space X. The CW-space is conjectured to be a classifying space for Artin groups and the conjecture is known to hold in many cases.

The formula for second homology has been implemented in the homological algebra software HAP [16]. Also, the π_1-equivariant cellular chain complex $C_*(\tilde{X})$ of the universal cover $\tilde{X}$ is implemented in HAP and can be used to compute integral cohomology rings or homology groups with twisted coefficients for particular Artin groups.

Our main aim is to investigate a generalization of the notion of Artin group (namely a *twisted Artin group*) with a view to obtaining polytopal classifying spaces, and corresponding explicit homological formulae, for a wider range of groups. Such results are needed to improve the functionality of the homological software HAP.

Our choice of generalization is influenced by the observation that Bieberbach groups (that is, fundamental groups of flat manifolds) admit polytopal classifying spaces. Indeed, the HAPCRYST software [16] implements a convex hull method for computing the fundamental domain for the action of a Bieberbach group π on Euclidean space $\mathbb{R}^n$. This domain induces a CW-structure on $\mathbb{R}^n$, and the resulting π-equivariant cellular chain complex $C_*(\mathbb{R}^n)$ is implemented and used for determining the cohomology of G. However, convex hull computations can be prohibitively expensive in high dimensions. It is desirable to achieve a theoretical and more easily implemented description of the polytopal fundamental domain whenever possible. For example, finitely generated free abelian groups are Bieberbach groups, but they are also Artin groups; the polytopal classifying space produced by HAPCRYST will coincide with the more easily computed classifying space for Artin groups.

Definition. A *twisted Artin group* is a finitely presented group with one relator

$$(xy)_m = (yx)_n$$

between each pair of distinct generators x, y where $|m - n| \in \{0, 2\}$ and $(m, n) \neq (1, 1)$. Here m, n depend on x, y and are either both positive integers or else both equal to ∞.

The fundamental groups of both the torus ($xy = yx$) and the Klein bottle ($x = yxy$) are twisted Artin groups. So too are all Artin groups, precisely four of the ten 3-dimensional Bieberbach groups (Proposition 5), the Baumslag-solitar group $B(1, 2)$ ($xy = yxyx$) and more generally the Baumslag-Solitar groups $B(k, k + 1)$ (Proposition 3).

Terminology and notation is explained in Section 2. Low dimensional homology of (non-twisted) Artin groups is considered in Section 3. We begin our investigation of twisted Artin groups in Section 4 with a description of the 2-generator case (Proposition 3) and an observation that these 2-generator groups admit 2-dimensional classifying spaces. We continue in Section 5 with a study of 3-generator twisted Artin groups. A comparison between the braid group $\langle x, y, z : xyx = yxy, yzy = zyz, xz = zx\rangle$ and twisted group $\langle x, y, z : xyxy = yx, yzyz = zy, xz = zx\rangle$ is inconclusive (Proposition

4) but demonstrates the delicate nature of the investigation. The 3-generator groups of *large type* are shown to admit 2-dimensional classifying spaces (Theorem 6). We note that all but one of the 3-generator groups of *small type* are Bieberbach groups and admit 3-dimensional classifying spaces; the exceptional group has torsion and thus does not admit a finite-dimensional classifying space (Proposition 5). In Section 6 we observe that a result of Appel and Schupp [2] can be routinely extended to yield examples of twisted Artin groups on more than three generators with 2-dimensional classifying spaces (Proposition 9).

In Section 7 we consider semi-direct products. A paper of C.T.C. Wall [23] explains how one can construct a free resolution for a group extension from free resolutions for the factor group and kernel group. This general construction is implemented in HAP. We explain in Theorem 12 how the construction can be made more explicit in some cases and lifted to the level of polytopal classifying spaces for certain semi-direct products. Theorem 12 is used to produce a polytopal classifying space for many *right-angled* twisted Artin groups (see Proposition 14). An implementation of Proposition 12 should improve HAP's efficiency in dealing with certain semi-direct products.

In the final Section we apply a technique from [17] to the group G generated by $x_1, \ldots, x_8$ subject to relations

$$\left.\begin{array}{lll}
(x_1x_2)_6 = (x_2x_1)_6, & (x_1x_4)_6 = (x_4x_1)_6, & x_1x_i = x_ix_1 \ \ (i \in [5,6,7,8]), \\
(x_2x_3)_3 = (x_3x_2)_3, & & x_2x_i = x_ix_2 \ \ (i \in [4,6,7,8]), \\
(x_3x_4)_3 = (x_4x_3)_3 & (x_1x_3)_3 = x_3, & x_3x_i = x_ix_3 \ \ (i \in [5,6,7,8]), \\
 & & x_4x_i = x_ix_4 \ \ (i \in [5,6,7,8]), \\
(x_5x_6)_3 = x_6, & (x_5x_7)_3 = x_7, & x_5x_8 = x_8x_5, \\
(x_6x_7)_4 = x_7x_6, & (x_6x_8)_5 = (x_8x_6)_3, & \\
(x_7x_8)_4 = (x_8x_7)_4. & &
\end{array}\right\} \quad (1)$$

Theorems 6, 12 and Proposition 9 are needed to establish an explicit classifying space for G. Furthermore, a HAP implementation of the space can be used to determine the following dimensions of its mod 2 cohomology groups.

$n =$	0	1	2	3	4	5	≥ 6
$\dim(H_n(G, \mathbb{Z}_2)) =$	1	5	14	21	18	6	0

The example thus illustrates the kind of functionality that the paper adds to HAP software.

We are grateful to the referee for many helpful comments.

2 Notation and terminology

We say that a relator $(xy)_m = (yx)_n$ is a *twisted Artin relator* if $|m-n| \in \{0, 2\}$ and $(m, n) \neq (1, 1)$, the case $m = n = \infty$ being allowed. We call $(m+n)/2$ the *weight* of the relator. If $m = n$ we say that the relator is *balanced*; otherwise we say the relator is *unbalanced*. In an unbalanced relator we say that x is the *heavy* generator if $m > n$ and that y is the *heavy* generator if $n > m$. A *twisted Artin group* A is

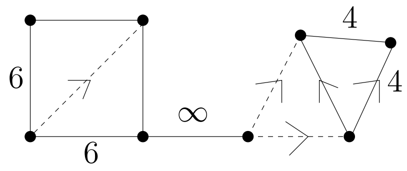


Figure 1:

thus a finitely presented group generated by a set of symbols $S = \{x_1, \ldots, x_n\}$ with one twisted Artin relator for every pair of distinct generators in S. A twisted Artin group is an *Artin group* if all relators are balanced.

To describe a twisted Artin group we define a graph D with: i) one vertex for each generator $x \in S$, ii) a solid edge between vertices $x, y \in S$ if they are related by a twisted Artin relator of weight ≥ 3, iii) a dotted edge between vertices $x, y \in S$ if they are related by an unbalanced twisted Artin relator of weight 2. A label m is assigned to an edge such that the corresponding relator is either $(xy)_m = (yx)_m$ or $(xy)_{m+1} = (yx)_{m-1}$. An unlabelled solid line means $m = 3$ while the unlabelled dotted line means $m = 2$. An arrow head is placed on all unbalanced edges pointing away from the heavy generator. We refer to D as a *Coxeter graph.* For example, the above group G defined by relators (1) is represented in Figure 1.

We associate a Coxeter group W_A to any twisted Artin group A by imposing the relations $x^2 = 1$ for $x \in S$. We also denote this Coxeter group by W_D where D is the graph defining A. The twisted Artin group is said to be of *finite type* if W_A is finite.

We say that an Artin or twisted Artin group is of *large type* if each pair of distinct generators is related by a defining relator of weight ≥ 3. We say that the group is of *small* type if each pair of distinct generators is related by a defining relator of weight 2. We say that the group is *right-angled* if each pair of distinct generators is related by a defining relator of weight 2 or ∞.

We say that a right-angled group is *directed* if all edges in the complete graph on S can be given directions, consistent with any directions in the Coxeter graph, such that the complete graph contains no directed cycles.

The *standard* 2*-complex* K associated to a group presentation is a CW-space with one 0-cell, one 1-cell for each generator in the presentation, and one 2-cell for each relator in the presentation. The 1-cells are oriented and labelled by generators. The attaching map of a 2-cell, when read in the appropriate direction from the appropriate starting point, spells the corresponding relator, inverses corresponding to a change in reading direction.

3 Second homology of Artin groups

Let D be the Coxeter graph of an Artin group (so there are no arrows or dotted edges). We associate integers p and q to D as follows. Let P denote the set of pairs $\{s, t\}$ of non-adjacent vertices in D. For two such pairs define $\{s, t\} \equiv \{s', t'\}$ if $t = t'$

and s is connected to s' by an odd labelled edge. This generates an equivalence relation on P. Say that an equivalence class is *torsion* if it is represented by a pair $\{s,t\}$ for which there exists a vertex v connected to both s and t by edges labelled with a 3. Then p is equal to the number of torsion equivalence classes in P. We set q_1 equal to the number of non-torsion equivalence classes in P. We set q_2 equal to the number of pairs of $\{s,t\}$ of vertices connected by an even labelled edge.

Let D^{odd} be the graph obtained from D by removing all edges labelled by an even integer or labelled by ∞. The graph D^{odd} has the same vertices as D. Let q_3 be the number of edges outside a maximal forest in D^{odd} (or, equivalently, the rank of the first integral homology $H_1(D^{\text{odd}},\mathbb{Z})$ of the graph). We set $q = q_1 + q_2 + q_3$.

Theorem 1 *Let A be an Artin group defined by a Coxeter graph D with vertex set S and associated Coxeter group W. Suppose that one or more of the following holds: (i) W is finite; (ii) W is an affine group of type $\tilde{A}_n$ or $\tilde{B}_n$; (iii) for every triple of generators $x,y,z \in S$ the three Artin relators $(xy)_k = (yx)_k$, $(yz)_l = (zy)_l$, $(xz)_m = (zx)_m$ are such that $\frac{1}{k}+\frac{1}{l}+\frac{1}{m} \leq 1$; (iv) every proper full subgraph of the Coxeter graph of W with no ∞ edges defines a Coxeter group satisfying (i), (ii) or (iii). Then the second integral homology of A is*

$$H_2(A,\mathbb{Z}) = (\mathbb{Z}_2)^p \oplus \mathbb{Z}^q$$

where p,q are the above integers associated to D.

The existence of such a simple formula for the second homology of Artin groups seems not to have been previously recorded in the literature. However, an analogous formula for the second homology of Coxeter groups has been observed by Howlett [18] using a very different approach. Our proof of Theorem 1 can be extended and used to recover Howlett's formula. To illustrate Theorem 1 we consider the affine braid group $\tilde{A}_n$ defined by the $(n+1)$-sided polygonal Coxeter graph with each edge labelled by 3 (Figure 2). The theorem implies:

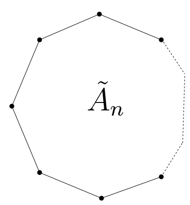


Figure 2:

$$\begin{aligned}
H_2(\tilde{A}_2,\mathbb{Z}) &= \mathbb{Z},\\
H_2(\tilde{A}_3,\mathbb{Z}) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z},\\
H_2(\tilde{A}_n,\mathbb{Z}) &= \mathbb{Z}_2 \oplus \mathbb{Z}, \quad n \geq 4.
\end{aligned}$$

We prove Theorem 1 by recalling the low-dimensional structure of a CW-space $X = B_D$ which is conjectured to be a classifying space for the Artin group A_D

associated to the Coxeter diagram D. The space B_D was introduced by M. Salvetti [21] but is also implicit in the earlier work of C. Squier [22]. The conjecture is known to hold in those cases covered by the theorem.

Let $\overline{x}$ be the image in W_D of the generator $x \in S \subset A_D$ and set $\overline{S} = \{\overline{x}: x \in S\}$. Assume for the moment that D is of finite type and let $n = |S|$. Then W_D can be realized as a group of orthogonal transformations of $\mathbb{R}^n$ with generators $\overline{x}$ equal to reflections [12]. Let $\mathcal{A}$ be the set of hyperplanes corresponding to all the reflections in W_D. For any point e in $\mathbb{R}^n \setminus \mathcal{A}$ we denote by P_D the convex hull of the orbit of e under the action of W_D. The face lattice of the n-dimensional convex polytope P_D depends only on the graph D. (To see this, first note that the vertices of P_D are the points $w \cdot e$ for $w \in W_D$ and that there is an edge between $w \cdot e$ and $w' \cdot e$ if and only if $w^{-1}w' \in \overline{S}$. Thus the combinatorial type of the 1-skeleton of P_D does not depend on the choice of point e. Furthermore, each vertex of the n-dimensional polytope P_D is incident with precisely n edges; hence P_D is simple and the face lattice of the polytope is determined by the combinatorial type of the 1-skeleton [3].)

Label each edge in P_D by the generating reflection $\overline{x} = w'w^{-1} \in \overline{S}$ determined by the edge's boundary vertices $w \cdot e, w' \cdot e$. Define the *length* of an element g in W_D to be the shortest length of a word in the generators representing it. It is possible to orient each edge in P_D so that its initial vertex gv and final vertex $g'v$ are such that the length of g is less than the length of g'. With this edge orientation the 1-skeleton coincides with the Hasse diagram for the weak Bruhat order on W_D. Each k-face in P_D has a least vertex in the weak Bruhat order. Reading the edge labels along the boundary of any 2-face, starting at the least vertex and and using edge orientations to determine exponents ± 1, yields a relator $(xy)_{m(i,j)}(yx)^{-1}_{m(i,j)}$ of the Artin group A_D. Furthermore, if F is any k-face of P_D, then $V_F = \{w \in W_D: w \cdot e \in F\}$ is a left coset of the *parabolic subgroup* $\langle T \rangle$ of W_D generated by some subset $T \subset \overline{S}$ of size $|T| = k$; this induces an isomorphism between the face lattice of P_D and the poset of cosets $\{w \cdot \langle T \rangle: T \subset \overline{S}, w \in W_D\}$ ordered by inclusion.

The above description of the polytope P_D is well-known. (We note that many authors prefer to deal with the dual polytope: since P_D is simple the dual is simplicial.)

The space B_D is obtained from the polytope P_D by isometrically identifying any two cells with similarly labelled 1-skeleta. More precisely, the group W_D acts cellularly on P_D. If a k-face F is mapped to a k-face F' under the action of $w \in W_D$, then there is a unique $w_0 \in W_D$ which maps F to F' in such a way that the least vertex of F maps to the least vertex of F'; we identify $w \cdot f$ with f for each point $f \in F$. Thus the face lattice of B_D is isomorphic to the poset of subsets of $\overline{S}$.

Suppose now that D is not of finite type. We define a subgraph D_i of D to be *maximal finite* if D_i is a full subgraph of D and D_i is of finite type. Let $D_1, \ldots, D_k$ be the list of maximal finite subgraphs of D. We denote by $D_i \cap D_j$ the full subgraph of D with vertices common to D_i and D_j. There is a canonical embedding of the polytope $P_{D_i \cap D_j}$ into the polytope P_{D_i}; such embeddings allow us to define P_D as the amalgamated sum of the polytopes $P_{D_1}, \ldots, P_{D_k}$. The space B_D is the connected space obtained from P_D by isometrically identifying any two cells with similarly labelled 1-skeleta; the identification is the unique one which respects orientations

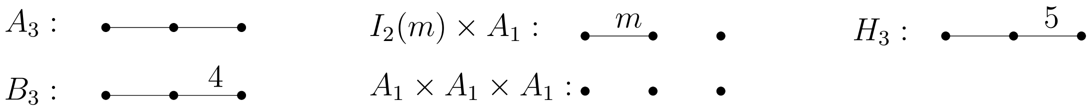


Figure 3:

of edges. The face lattice of the space B_D is isomorphic to the poset $S^f = \{T \subset \overline{S} \colon |\langle T \rangle| < \infty\}$ ordered by inclusion.

The space B_D clearly has fundamental group isomorphic to the Artin A_D. The so called $K(\pi, 1)$ *conjecture for Artin groups* implies that the homotopy groups $\pi_i(B_D)$ are trivial for all $i \geq 2$. The conjecture is known to hold in a number of cases, including:

1. if D is of finite type [22, 21].
2. if for every triple of generators $a, b, c \in S$ the three Artin relators $(ab)_k = (ba)_k$, $(bc)_l = (cb)_l$, $(ac)_m = (ca)_m$ are such that $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \leq 1$. [2]
3. if D is of affine type $\tilde{A}_n$ [10] or affine type $\tilde{B}_n$ [7].
4. if every proper full subgraph of D with no ∞ edges represents an Artin group for which the conjecture is known to hold [17]. (See also [8] for a special case of this.)

Let us set $G = A_D$. Let $\tilde{B}_D$ be the universal cover of B_D and let $C_*(\tilde{B}_D)$ be the cellular chain complex of the CW-space $\tilde{B}_D$. Thus $C_*(\tilde{B}_D)$ is a free $\mathbb{Z}G$-chain complex with precisely one generator in dimension 0 which we denote by []. The cellular chain complex has precisely n generators $[x]$ $(x \in S)$ in dimension 1. The $\mathbb{Z}G$-module $C_2(\tilde{B}_D)$ is freely generated by unordered pairs $[x, y]$ of distinct generators in S with edge label $m(x, y) \neq \infty$. The $\mathbb{Z}G$-module $C_3(\tilde{B}_D)$ is freely generated by unordered triples $[x, y, z]$ of distinct generators in S for which the full subgraph of D is one of the five in Figure 3 (representing all possible 3-generator finite type Artin groups [19]).

The first three boundary homomorphisms of $C_*(\tilde{B}_D)$ can be derived from the 3-dimensional polytopes in Figure 4 and are given in Table 1. The cellular chain complex $C_*(\tilde{B}_D)$ is implemented in the computational algebra package HAP [16] and the boundary homomorphisms in Table 1 can also be obtained from this package.

The second homology $H_2(B_D, \mathbb{Z})$ is the second homology of the chain complex $C_*(\tilde{B}_D) \otimes_{\mathbb{Z}G} \mathbb{Z}$ and can be obtained from the boundary maps in Table 1 by setting $x = y = z = 1$. In cases where the $K(\pi, 1)$-conjecture holds we have $H_2(G, \mathbb{Z}) = H_2(B_D, \mathbb{Z})$ and Theorem 1 is a consequence of the following.

Proposition 2 *For any Coxeter graph D we have*

$$H_2(B_D, \mathbb{Z}) = (\mathbb{Z}_2)^p \oplus \mathbb{Z}^q$$

where p, q are the integers associated to D above.

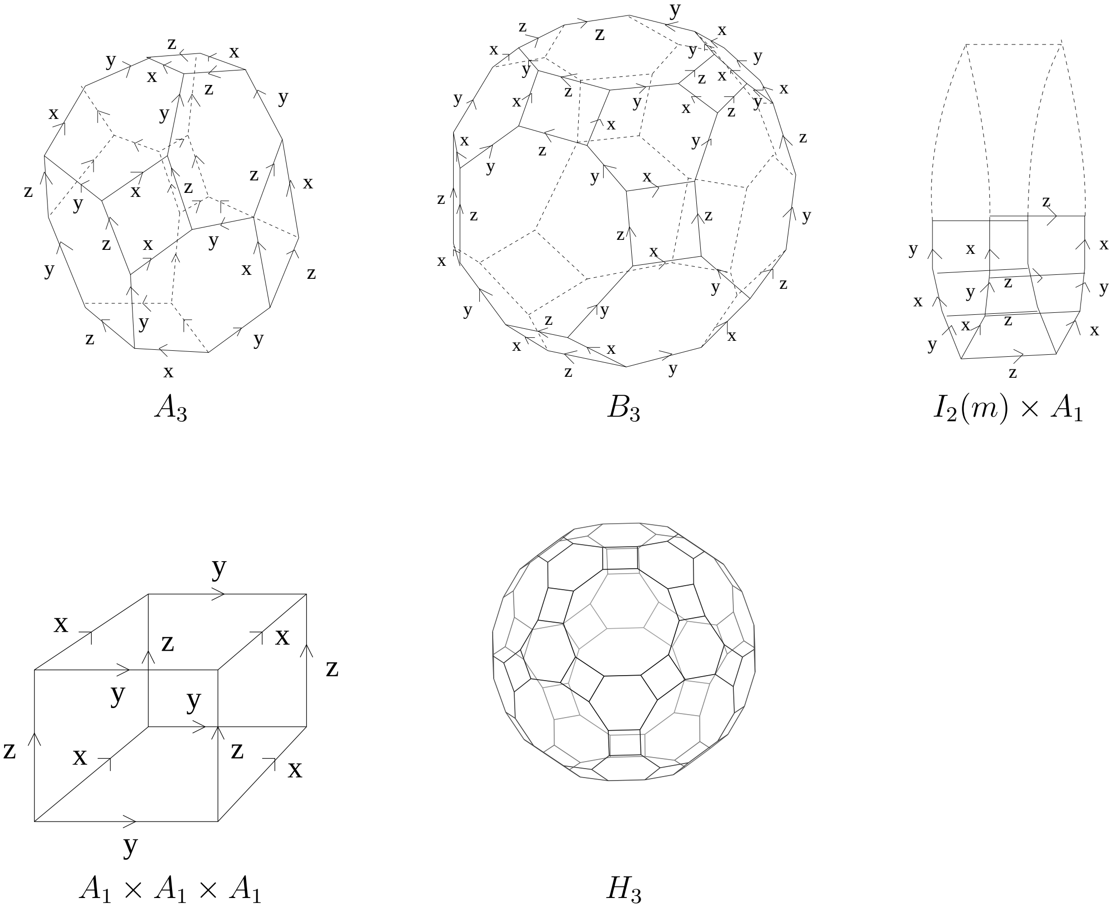


Figure 4:

Generator	Boundary	Generator type
$[x]$	$[] - x[]$	
$[x, y]$	$(1 + xy + \cdots (xy)^k)[x] + (x + xyx + \cdots (xy)^{k-1}x)[y]$ $-(1 + yx + \cdots (yx)^k)[y] - (y + yxy + \cdots (yx)^{k-1}y)[x]$	$m(x, y) = 2k + 1$
$[x, y]$	$(1 + xy + \cdots (xy)^{k-1})[x] + (x + xyx + \cdots (xy)^{k-1}x)[y]$ $-(1 + yx + \cdots (yx)^{k-1})[y] - (y + yxy + \cdots (yx)^{k-1}y)[x]$	$m(x, y) = 2k$
$[x, y, z]$	$(1 - z + yz - xyz)[x, y] + (1 - x + yx - zyx)[y, z]$ $-(1 - y - xzy + zy + yxzy + xy)[x, z]$	A_3
$[x, y, z]$	$(1 - z + yz - zyz - xyz + zxyz - yzxyz + zyzxyz)[x, y]$ $+(1 - x + yx - zyx + yzyx - xyzyx)[y, z]$ $-(1 - y + xy + zy - zxy - yzy + yzxy + xyzy$ $-xyzxy - zyzxy + zxyzxy - yzxyzxy)[x, z]$	B_3
$[x, y, z]$	$(1 - z + yz - xyz - zyz + zxyz + yzyz - yzxyz$ $-xyzyz + xyzxyz + zyzxyz - zxyzxyz$ $-yzyzxyz + yzxyzxyz + xyzyzxyz$ $-xyzxyzxyz - zyzxyzxyz + zxyzxyzxyz$ $-yzxyzxyzxyz + zyzxyzxyzxyz)[x, y]$ $+(1 - x + yx - zyx + yzyx - xyzyx - zyzyx$ $+zxyzyx - yzxyzyx + zyzxyzyx$ $-yzyzxyzyx + xyzyzxyzyx)[y, z]$ $-(1 - y + xy + zy - zxy - yzy + yzxy + xyzy$ $+zyzy - xyzxy - zyzxy - zxyzy + zxyzxy$ $+yzyzxy + yzxyzy - yzxyzxy - xyzyzxy$ $-zyzxyzy + xyzxyzxy + zyzxyzxy$ $+yzyzxyzy - zxyzxyzxy - yzyzxyzxy$ $-xyzyzxyzy + yzxyzxyzxy + xyzyzxyzxy$ $-xyzxyzxyzxy - zyzxyzxyzxy + zxyzxyzxyzxy$ $-yzxyzxyzxyzxy)[x, z]$	H_3
$[x, y, z]$	$(1 - z)[x, y] - (1 + xy + \cdots (xy)^k)[x, z]$ $+(y + yxy + \cdots (yx)^{k-1}y)[x, z]$ $-(x + xyx + \cdots (xy)^{k-1}x)[y, z]$ $+(1 + yx + \cdots (yx)^k)[y, z]$	$I_2(m)$ $m(x, y) = 2k + 1$
$[x, y, z]$	$(1 - z)[x, y] - (1 + xy + \cdots (xy)^{k-1})[x, z]$ $+(y + yxy + \cdots (yx)^{k-1}y)[x, z]$ $-(x + xyx + \cdots (xy)^{k-1}x)[y, z]$ $+(1 + yx + \cdots (yx)^{k-1})[y, z]$	$I_2(m)$ $m(x, y) = 2k$
$[x, y, z]$	$(1 - z)[x, y] + (1 - x)[y, z] + (y - 1)[x, z]$	$A_1 \times A_1 \times A_1$

Table 1:

Proof Observe that, after tensoring with the integers, the boundary homomorphisms in Table 1 are determined by the following formulae on generators: $d_2([x,y]) = 0$ if $m(x,y)$ is even, $d_2([x,y]) = [y] - [x]$ if $m(x,y)$ is odd, $d_3([x,y,z]) = 2[x,z]$ for a corresponding Coxeter diagram $D = A_3$, $d_3([x,y,z]) = [x,z] - [y,z]$ for a corresponding Coxeter diagram $I \times A_2(m)$ with m odd, and $d_3([x,y,z] = 0$ otherwise. It is an easy exercise to show that these formulae lead to the following definition of the integers p and q in the theorem. $\square$

We know of no nice formula for the third integral homology of B_D. Nevertheless, the HAP [16] implementation of the $\mathbb{Z}A_D$-equivariant cellular chain complex $C_*(\tilde{B}_D)$ can be used to compute homology of specific groups. For example, the formulae

$$\begin{array}{ll} H_3(\tilde{A}_n, \mathbb{Z}) = 0, & n = 3 \\ H_3(\tilde{A}_n, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}, & n = 4 \\ H_3(\tilde{A}_n, \mathbb{Z}) = \mathbb{Z}, & n = 5 \\ H_3(\tilde{A}_n, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}, & n = 6 \\ H_3(\tilde{A}_n, \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}, & n \geq 7 \end{array}$$

for the affine braid group $\tilde{A}_n$ can be verified for a range of values of n using HAP. A proof of these formulae in general is given by the first author in [13].

4 Twisted Artin groups on two generators

We now consider the general 2-generator twisted Artin group

$$T(m,n) = \langle x, y : (xy)_m = (yx)_n \rangle.$$

The group $T(2,2)$ is free abelian group of rank two. The group $T(\infty,\infty)$ is free of rank two. The group $T(1,3)$ is the fundamental group of the Klein bottle. The group $T(2,4)$ is the Baumslag-Solitar group $B(1,2)$. The group $T(3,3)$ is the 3-string braid group. We thank the referee for pointing out the following general description of 2-generator unbalanced twisted Artin groups.

Proposition 3 *If $m \geq 2$ is even then $T(m-1, m+1)$ is a semi-direct product $F \times_\alpha C_\infty$ of an infinite cyclic group C_∞ acting on a free group F of rank $m/2$. If $m \geq 3$ is odd then $T(m-1, m+1)$ is the Baumslag-Solitar group $B(k, k+1)$ with $k = (m-1)/2$.*

Proof. Suppose that $m \geq 2$ is even and set $k = m/2$. Applying the Tietze transformation $t = xy$ yields $T(m-1, m+1) = \langle x, y, t : t = xy, t^{k-1}x = yt^k \rangle$. The Tietze transformation $x = ty^{-1}$ then gives $T(m-1, m+1) = \langle y, t : t^k y^{-1} = yt^k \rangle$ which is a semi-direct produce of the infinite cyclic subgroup $C_\infty = \langle t \rangle$ with the normal subgroup $F = \langle y, t^{-1}yt, \ldots, t^{-k+1}yt^{k-1} \rangle$. This group F is free of rang k.

Suppose now that $m \geq 3$ is odd and set $k = (m-1)/2$. Applying the Tietze transformation $t = xy$ yields $T(m-1, m+1) = \langle x, y, t : t = xy, t^k = yt^k x \rangle$. The Tietze transformation $x = ty^{-1}$ then gives $T(m-1, m+1) = \langle y, t : t^k = yt^{k+1}y^{-1} \rangle$ which is the definition of $B(k, k+1)$. $\square$

Note that the standard 2-complex associated to the twisted Artin presentation for $T(m, n)$ is a classifying space by a result of Chiswell [11] which asserts that a one-relator presentation is aspherical if the relator is not a proper power.

5 Twisted Artin groups on three generators

All 3-generator Artin groups admit finite-dimensional classifying spaces: for those Artin groups not of finite type the standard 2-complex associated to the presentation is known to be aspherical [2]; for those groups of finite type a classifying space is obtained by adding a single cell to the 2-complex as explained in Section 3. The construction of finite-dimensional classifying spaces for 3-generator twisted Artin groups is more delicate. Consider the 3-generator twisted Artin groups G_1, G_2, G_3 represented in Figure 5.. The group G_1 is an Artin group of finite type and

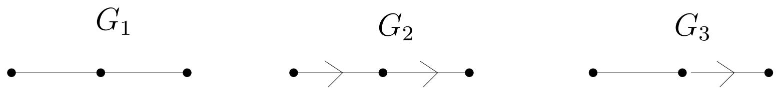


Figure 5:

thus admits a 3-dimensional classifying space. We do not know if G_2 admits a 3-dimensional classifying space, though we do know that the associated 2-complex is not aspherical.

Proposition 4 *The 2-complex K associated to the presentation for G_2 has a non-trivial second homotopy group $\pi_2 K$.*

Proof. A homotopically non-trivial map $S^2 \to K$ is shown in Figure 6. □

The proof of Proposition 4 was found using a GAP implementation of the *word-reversing algorithm* described in [14]. The word-reversing algorithm is known to terminate for all Artin groups of finite type. Experimentation suggests that the algorithm also terminates for the twisted Artin group G_2. However, experimentation suggests that the algorithm does not terminate in general for the group G_3 and we do not know if the 2-complex for this group has non-trivial second homotopy group.

We can say more in the case of small and large twisted Artin groups. In the following two subsections we construct finite-dimensional classifying spaces for all but one of the small 3-generator twisted Artin groups and for all large 3-generator twisted Artin groups.

5.1 Small 3-generator groups

Consider the Klein bottle group $Q = \langle s, t | sts = t \rangle$ and the infinite cyclic group $N = \langle x | \ \rangle$. Using the group action $\alpha: Q \to \mathrm{Aut}(N)$ defined by $\alpha(s)(x) = {}^s x = x^{-1}$, $\alpha(t)(x) = {}^t x = x^{-1}$ form the semi-direct product $G = N \times_\alpha Q$; this is a twisted Artin group represented in Figure 7. The standard classifying spaces for Q and

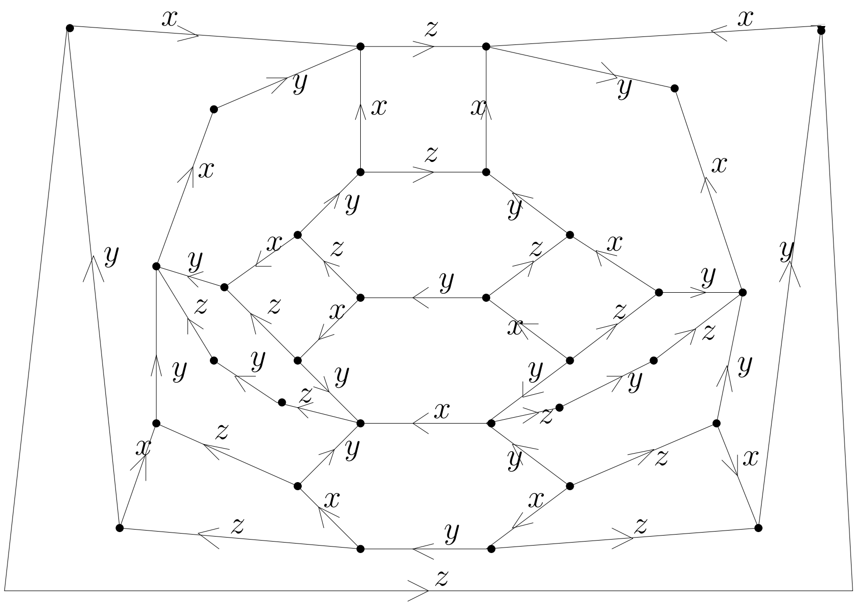


Figure 6:

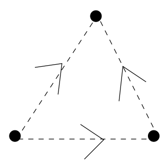


Figure 7:

N can be obtained from the labelled unit square $P(Q)$ and labelled unit interval $P(N)$ (Figure 8) by identifying vertices and similarly labelled edges, identifications respecting edge orientations. We denote these classifying spaces by B_Q and B_N. Let B_G be the space obtained from the labelled unit cube $P(G)$ (Figure 9) by identifying similarly labelled facets. Thus B_G has one vertex e^0, three 1-cells e^1_x, e^1_s, e^1_t, three 2-cells $e^2_{s,t}, e^2_{x,s}, e^2_{x,t}$ and one 3-cell $e^3_{x,s,t}$. By van Kampen's theorem the fundamental group $\pi_1 B_G$ has presentation $\langle x, s, t | sts = t, sxs^{-1} = x^{-1}, txt^{-1} = t^{-1}\rangle$ and is thus isomorphic to G. It is instructive to give two explanations of why B_G is a classifying space for G.

Let $\tilde{B}_G$ be the universal cover of B_G. Since $\pi_n B_G = \pi_n \tilde{B}_G$ for $n \geq 2$, and since the first non-vanishing homotopy group of $\tilde{B}_G$ is isomorphic to the first non-vanishing integral homology group, it suffices to show that the cellular chain complex $C_*(\tilde{B}_G)$ is acyclic. For each cell e in B_G we choose one covering cell $\tilde{e}$ in $\tilde{B}_G$. The cellular chain complex $C_*(\tilde{B}_G)$ is a chain complex of free $\mathbb{Z}G$-modules with one free summand $\mathbb{Z}G\,\tilde{e}$ for each cell e in B_G. This chain complex can be viewed as the total complex

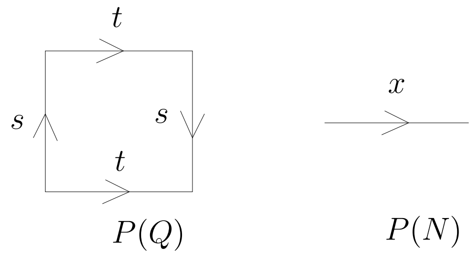


Figure 8:

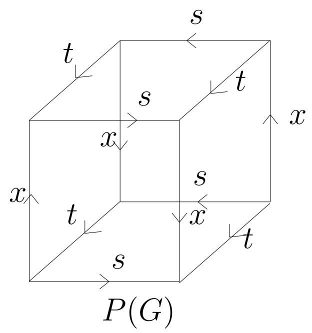


Figure 9:

of the following bicomplex D_{**} of free $\mathbb{Z}G$-modules

$$\begin{tikzcd}
\mathbb{Z}G\,\tilde{e}^3_{x,s,t} \arrow[r, "d_h"] \arrow[d, "d_v"] & \mathbb{Z}G\,\tilde{e}^2_{x,s} \oplus \mathbb{Z}G\,\tilde{e}^2_{x,t} \arrow[r, "d_h"] \arrow[d, "d_v"] & \mathbb{Z}G\,\tilde{e}^1_x \arrow[d, "d_v"] \\
\mathbb{Z}G\,\tilde{e}^2_{s,t} \arrow[r, "d_h"] & \mathbb{Z}G\,\tilde{e}^1_s \oplus \mathbb{Z}G\,\tilde{e}^1_t \arrow[r, "d_h"] & \mathbb{Z}G\,\tilde{e}^0
\end{tikzcd}$$

in which the boundary homomorphisms can be read directly from the labelled polytope $P(G)$. To do this we orient each edge, facet and unit cube itself, giving similarly labelled edges and facets the same orientation. For a given choice of orientations the homomorphisms are defined in Table 2. Let $C_*(\tilde{B}_N)$ be the cellular chain complex of the universal cover of $B_N = S^1$. The $\mathbb{Z}G$-chain complex $R_* = C_*(\tilde{B}_N) \otimes_{\mathbb{Z}N} \mathbb{Z}G$ is a free $\mathbb{Z}G$-resolution of the $\mathbb{Z}G$-module $\mathbb{Z}Q$. Note that the first and last columns in the bicomplex D_{**} are isomorphic to R_*, and the middle column is isomorphic to $R_* \oplus R_*$. So the bicomplex D_{**} is such that each column is a resolution of the $\mathbb{Z}G$-module $H_0(D_{n*}) = C_n(\tilde{B}_Q)$, where B_Q is the 2-dimensional classifying space constructed from the presentation $Q = \langle s, t | sts = t \rangle$. Since $C_*(\tilde{B}_Q)$ is acyclic, it follows that the total complex of D_{**} is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Hence B_G is a classifying space for G.

An alternative explanation of the asphericity of B_G is based on the observation that the automorphism $\alpha(q)\colon N \to N, n \mapsto {}^q n$ is induced by a map $\tilde{\alpha}(q)\colon \tilde{B}_N \to \tilde{B}_N, x \mapsto {}^{\tilde{\alpha}(q)}x$ which permutes faces for each $q \in Q$. We denote the canonical action

$$
\begin{aligned}
d_v(\tilde{e}^1_x) &= (x-1)\tilde{e}^0\\
d_h(\tilde{e}^1_s) &= (s-1)\tilde{e}^0\\
d_h(\tilde{e}^1_t) &= (t-1)\tilde{e}^0\\
d_v(\tilde{e}^2_{x,s}) &= (x-1)\tilde{e}^1_s\\
d_v(\tilde{e}^2_{x,t}) &= (t^{-1}-xt^{-1})\tilde{e}^1_t\\
d_h(\tilde{e}^2_{x,s}) &= (1+xs)\tilde{e}^1_x\\
d_h(\tilde{e}^2_{x,t}) &= (1+xt^{-1})\tilde{e}^1_x\\
d_h(\tilde{e}^2_{s,t}) &= (-1-t^{-1}s^{-1})\tilde{e}^1_s + (t^{-1}s^{-1}-t^{-1})\tilde{e}^1_t\\
d_v(\tilde{e}^3_{x,s,t}) &= (x-1)\tilde{e}^2_{s,t}\\
d_h(\tilde{e}^3_{x,s,t}) &= (1+xt^{-1})\tilde{e}^2_{x,s} + (-1-xs)\tilde{e}^2_{x,t}
\end{aligned}
$$

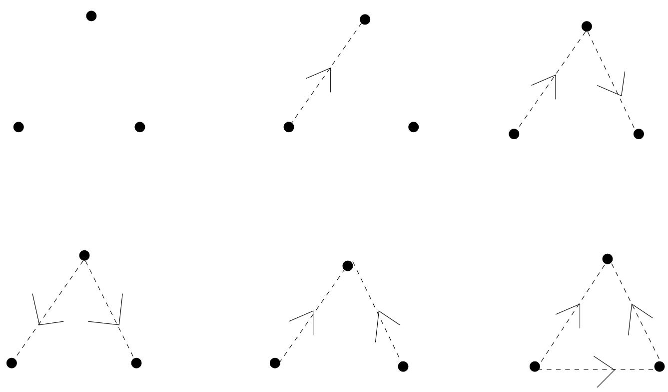
Table 2:

of $n \in N$ on $a \in B_N$ by na. The direct product $\tilde{B}_N \times \tilde{B}_Q$ is contractible since both spaces $\tilde{B}_N$, $\tilde{B_Q}$ are contractible. The semi-direct product $N \times_\alpha Q$ acts on this direct product by

$$^{(n,q)}(a,b) = (n\,{}^{\tilde{\alpha}(q)}a, q\,b)$$

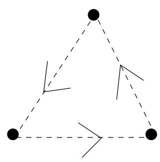
for $(n,q) \in N \times_\alpha G$ and $(a,b) \in \tilde{B}_N \times \tilde{B}_Q$. The action is free and B_G can be viewed as the quotient of $\tilde{B}_N \times \tilde{B_Q}$ by the action. Hence $\mathbb{R}^3 \cong \tilde{B}_N \times \tilde{B_Q}$ is the universal covering space of B_G and so B_G is aspherical. Furthermore, B_G is a flat manifold.

Proposition 5 *The six small twisted Artin groups*



represent four distinct isomorphism classes of groups, each of which is a 3-dimensional Bieberbach group with a classifying space obtained by suitably identifying faces of the

3-cube. The one remaining small twisted Artin group



does not admit a finite-dimensional classifying space.

Proof. The preceding argument proves that the sixth group G in the proposition is Bieberbach with cubical fundamental domain. A similar argument applies to the first five groups. It is routine to show that the first six groups represent four isomorphism classes. For the seventh group in the proposition one can readily construct a faithful crystallographic action on $\mathbb{R}^3$. In fact, using the computer algebra system GAP, this seventh group can be identified as $SpaceGroup(3, 62)$ in GAP's library of crystallographic groups. As it is not one of the ten 3-dimensional Bieberbach groups it must have torsion, and thus can not admit a finite-dimensional classifying space. □

5.2 Large 3-generator groups

Theorem 6 *Let A be a 3-generator twisted Artin group of large type. Then the 2-complex associated to the presentation of A is a classifying space $K(A, 1)$.*

Proof. Consider the presentation

$$P = \langle x, y, z : (xy)_l = (yx)_{l'}, (yz)_m = (zy)_{m'}, (xz)_n = (zx)_{n'} \rangle$$

of a large twisted Artin group A, with $l \geq l', m \geq m', n \geq n'$. Applying the Tietze transformation $a = xy, b = yz, c = zx$ we get a new presentation

$$P' = \langle x, y, z, a, b, c : a = xy, b = yz, c = zx, a^r x^{\epsilon_1} = y a^{r'} x^{\epsilon_1'}, b^s y^{\epsilon_2} = z b^{s'} y^{\epsilon_2'}, c^t z^{\epsilon_3} = x c^{t'} z^{\epsilon_3'} \rangle$$

with $r \geq r', s \geq s', t \geq t', \epsilon_i, \epsilon_i' \in \{0, 1\}$ and $\epsilon_i + \epsilon_i' = 1$. (For example, if

$$P = \langle x, y, z \mid xyxyx = yxyxy, \ yzyz = zyzy, \ zxz = xzx \rangle$$

then

$$P' = \langle x, y, z, a, b, c \ : \ xy = a, \ yz = b, \ zx = c, \ a^2 x = ya^2, \ b^2 = zby, \ cz = xc \rangle. \ )$$

Let K and K' be the 2-complexes associated to the presentations P and P'. The 1-cells of K' are oriented and labelled by the generators x, y, z, a, b, c. There are six 2-cells in K' whose attaching maps spell the six relators of P', inverses corresponding to a change in reading direction. We refer to any two consecutive edges in the boundary of a 2-cell as a *corner* of the 2-cell or of the corresponding relator.

The Tietze equivalence between presentations corresponds to a homotopy equivalence between K and K'. To establish the theorem we need only show that $\pi_2 K =$

$\pi_2 K' = 0$ (since Hurewicz's theorem will then imply that $\pi_n K' = H_n(K', \mathbb{Z}) = 0$ for $n \geq 2$). A standard technique is to consider the *star graph* of K'. This graph has one vertex for each generator and one vertex for the inverse of each generator in the presentation P'. There is a single undirected edge between vertices u and v^{-1} if the word uv corresponds to precisely one corner of a relator. There are two undirected edges between vertices u and v^{-1} if the word uv corresponds to more than one corner of a relator.

If the weight of each twisted Artin relator in P' is at least 5 and less than ∞ then the star graph of K' is that of Figure 10. If one or more of the relators have weight less than 5 or equal to ∞ then the star graph of P' is a subgraph of that shown in Figure 10. In Figure 10 the letters α, β, γ assigned to edges of the graph correspond

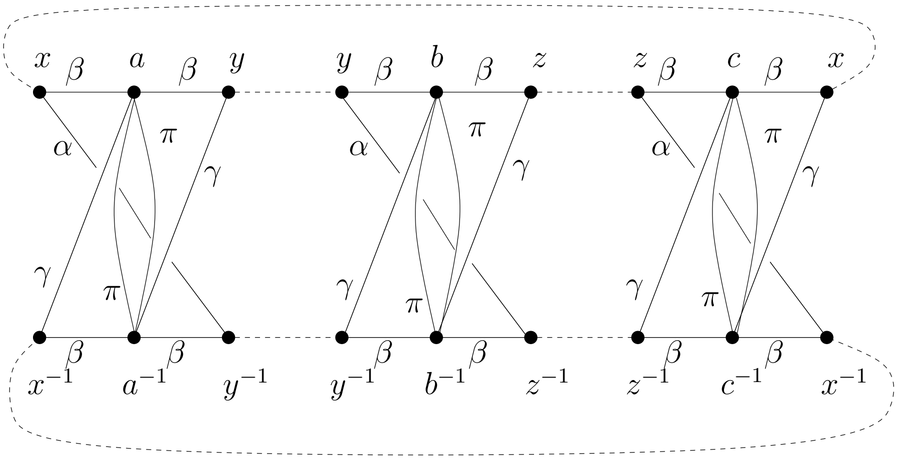


Figure 10:

to the "angles" assigned to corners of the 2-cells of K' as illustrated in Figure 11. Set $\alpha = 5\pi/8, \beta = 3\pi/8, \gamma = 2\pi/8$ radians and note that: (i) no loop in the star graph has edge angles summing to less than 2π; (ii) that the angles in each relator disc (Figure 11) sum to $(n-2)\pi$. We can conclude that K' is aspherical as follows. If K' were not aspherical then some non-trivial homotopy class in $\pi_2(K')$ would induce a tessellation of the 2-sphere by relator discs; the "angles" around each vertex in the tessellation sum to at least 2π; using the fact that the "angles" in each relator disc sum to $(n-2)\pi$ the sphere would have to have Euler characteristic different to 2, which is impossible. This proves the theorem. $\square$

Digressing briefly from our main theme, we remark that if all relators in P' are balanced then the cells of K' can be given euclidean metrics (illustrated visually in Figure 11) that agree on overlaps and which induce the angles $\alpha = 5\pi/8, \beta = 3\pi/8, \gamma = 2\pi/8$. Our proof of Theorem 6 thus provides an alternative proof of the following theorem. (See for instance [6] for a definition of *non-positive curvature* in this context.)

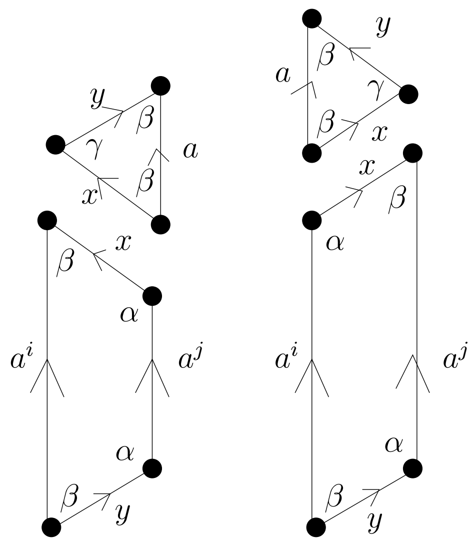


Figure 11:

Theorem 7 *[5] Let A be a 3-generator Artin group of large type. Then the standard 2-complex associated to A is homotopy equivalent to a non-positively curved classifying space $K(A,1)$.*

Proposition 8 *Let A be a 3-generator twisted Artin group whose Coxeter graph has one or more ∞ edges. Then the standard 2-complex associated to A is homotopy equivalent to a non-positively curved classifying space $K(A,1)$.*

Proof. Consider the presentation $P = \langle x, y, z : (xy)_l = (yx)_{l'}, (yz)_m = (zy)_{m'} \rangle$. Applying the Tiezte transformation $a = xy, b = yz$ yields an equivalent presentation $P' = \langle x, y, z, a, b : a = xy, b = yz, a^r x^{\epsilon_1} = ya^{r'} x^{\epsilon'_1}, b^s y^{\epsilon_2} = zb^{s'} y^{\epsilon'_2} \rangle$. Then eliminating x and z yields the equivalent presentation $P'' = \langle y, a, b : a^r (ay^{-1})^{\epsilon_1} = ya^{r'} (ay^{-1})^{\epsilon'_1}, b^s y^{\epsilon_2} = (y^{-1}b) b^{s'} y^{\epsilon'_2} \rangle$. Since $\epsilon_i + \epsilon'_i = 1$ the 2-cells in the 2-complex of P'' can be given the metric of a rectangle so that K'' is non-positively curved. $\square$

6 Some 2-dimensional groups on n generators

The hypothesis of Theorem 6 excludes any twisted Artin relators of weight 2. However, we have the following proposition which is a routine generalisation of a result of Appel and Schupp [2] for Artin groups. The hypothesis of the proposition excludes any unbalanced twisted Artin relators of weight greater than 2.

Proposition 9 *Suppose that a twisted Artin group A is such that all relators of weight $m \geq 3$ are balanced. Suppose also that for every triple of generators $x, y, z \in S$ the defining relators $(xy)_{k'} = (yx)_{k''}$, $(yz)_{l'} = (zy)_{l''}$, $(xz)_{m'} = (zx)_{m''}$ have weights k, l, m such that $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \leq 1$. Then the standard 2-complex associated to A is a classifying space $K(A,1)$.*

In the free group on $S = \{x_1, \ldots, x_n\}$ any word w can be uniquely written in reduced form

$$w = x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_k}^{e_k}$$

with $x_{i_j} \neq x_{i_{j+1}}$ and $e_j \in \mathbb{Z}$. We call each $x_{i_j}^{e_j}$ a *syllable* and say that w has k syllables. The following lemma was proved in [2]

Lemma 10 *[2] Let N be the normal closure of the word $(xy)_m(yx)_m^{-1}$ in the free group on x, y. Any non-trivial word $w \in N$ contains at least $2m$ syllables.*

This lemma was used in [2] to prove Proposition 9 for Artin groups. An outline of the argument is as follows. Suppose that A is an Artin group with generating set S and with relations $(xy)_m = (yx)_m$ satisfying the hypothesis of Proposition 9. Let K be the standard 2-complex associated to the presentation of A. So K has one 0-cell, one 1-cell for each generator in S, and one 2-cell for each relator $(xy)_m = (yx)_m$ with $m \neq \infty$. The fundamental group $\pi_1 K$ is certainly isomorphic to A. If we prove that the second homotopy group $\pi_2 K$ is trivial, then $\pi_k K = 0$ for $k \geq 2$ by the Hurewicz Theorem. Now any non-trivial element in $\pi_2 K$ is represented by a homotopically non-trivial map from the 2-sphere $\phi\colon S^2 \to K$. The CW-structure on K induces a CW-structure on S^2 in which cells correspond to the relators of A. In particular the cells are polygons with oriented edges labelled by generators of A. The cells can be partitioned into simply connected regions of S^2 in such a way that all edges in any given region are labelled by just two generators, and the edges in any two neighbouring regions are labelled by precisely three generators. Suppose that a region's edges are labelled by two generators x, y. Then the boundary of this region spells a word w in the normal closure of the word $(xy)_m(yx)_m^{-1}$. Using the fact that one-relator groups (whose relator is not a proper power) are aspherical we can assume that the word w is non-trivial. By Lemma 10 this boundary word w has at least $2m$ syllables. We now give S^2 a new CW-structure in which the 1-cells correspond to the syllables in the boundaries of the regions; by construction any vertex is incident with at least three 1-cells. An element of $\pi_2 K$ thus gives rise to a tessellation of S^2 by polygonal regions with $2m$ sides, and with each vertex degree at least 3. We give each region the metric of a regular euclidean polygonal disc with $2m$ sides. Using the hypothesis of Proposition 9 we observe that the sum of the angles subtended by any vertex is at least 2π. Hence we have endowed S^2 with a metric of non-positive curvature. As this is not possible there can be no non-trivial elements in $\pi_2 K$.

The following lemma extends this argument to a proof of Proposition 9.

Lemma 11 *Let N be the normal closure of the word $xyxy^{-1}$ in the free group on x, y. Any non-trivial word $w \in N$ contains at least 4 syllables.*

Proof. Let K be the 2-complex associated to the presentation of $A = \langle x, y | xyxy^{-1} \rangle$. Then K is the Klein bottle and the covering map $\mathbb{R}^2 \to K$ induces a tessellation of the plane $\mathbb{R}^2$ with square fundamental domain. The 1-skeleton of this tessellation is the Cayley graph of A with edges labelled by generators x and y. All horizontal edges are labelled by x and all vertical edges are labelled by y. Any non-trivial word $w \in N$ corresponds to a loop in this Cayley graph. Any such loop must involve at least two vertical edges and two horizontal edges. Thus the loop must involve at least four syllables. □

7 Some finite-dimensional groups on n generators

Our aim in this section is to generalize the first statement of Proposition 5. The starting point is a result of C.T.C. Wall [23] which states that, given a group extension $1 \to N \to G \to Q \to 1$, a free $\mathbb{Z}N$-resolution R_*^N of $\mathbb{Z}$ and free $\mathbb{Z}Q$-resolution R_*^Q of $\mathbb{Z}$, there is a free $\mathbb{Z}G$-resolution $R_*^G = R_*^N \tilde{\otimes} R^Q$ where the free $\mathbb{Z}G$-rank of R_n^G is $\mathrm{rank}_{\mathbb{Z}G}(R_n^G) = \sum_{p+q=n} \mathrm{rank}_{\mathbb{Z}N}(R_p^N) \times \mathrm{rank}_{\mathbb{Z}Q}(R_q^Q)$.

We say that the resolution R_*^Q is of *dimension* at most D and write $\dim(R_*^Q) \le D$ if $\mathrm{rank}_{\mathbb{Z}Q}(R_q^Q) = 0$ for all $q > D$. An analysis of Wall's resolution readily shows that:

1. If $\dim(R_*^N) \le D_N$ and $\dim(R_*^Q) \le D_Q$ then $\dim(R_*^G) \le D_N \times D_Q$.
2. If $\dim(R_*^N) \le D_N$ and R_*^Q is periodic of period c then the resolution R_*^G is periodic of period c in degrees higher than D_N.

These two observations are particularly useful for studying certain crystallographic groups. For instance, if an n-dimensional crystallographic group G has point group Q with periodic homology of period c, then the second observation implies that the homology of G can be determined in all degrees simply by calculating it up to degree $n + c$. As an example consider the group $G = SpaceGroup(3, 144)$ from GAP's crystallographic group library [15]. The following commands (involving the HAP package [16])

```
gap> G:=SpaceGroup(3,144);;
gap> IsCyclic(PointGroup(G));
true
gap> List([1..6],n->GroupHomology(G,n));
[ [ 2, 6 ], [ 3, 0 ], [ 2, 2, 6, 6 ], [ 3 ], [ 2, 2, 6, 6 ], [ 3 ] ]
```

establish

$$\begin{array}{lcl} H_1(G,\mathbb{Z}) & = & \mathbb{Z}_2 \oplus \mathbb{Z}_6, \\ H_2(G,\mathbb{Z}) & = & \mathbb{Z}_3 \oplus \mathbb{Z}, \\ H_{2k+1}(G,\mathbb{Z}) & = & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_6, (k \ge 1) \\ H_{2k}(G,\mathbb{Z}) & = & \mathbb{Z}_3, (k \ge 2). \end{array}$$

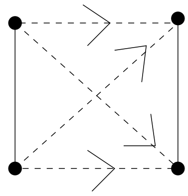
In order to make this kind of computation more efficient (the above took 2 seconds) we need explicit descriptions of Wall's resolution in special cases, rather than the recursive description for the general case implemented in [16]. For the special case when the group Q is cyclic of prime-power order and G is a semi-direct product of Q with a lattice N, the main result of [1] provides such a description. In this section we focus on the special case when the group G is a semi-direct product of two Bieberbach groups Q and N.

Let us review the example in Section 5.1 of the semi-direct product $G = N \times_\alpha Q$ of the infinite cyclic group N and Klein bottle group Q. The essential ingredients of this example are two groups N, Q with known classifying spaces B_N, B_Q, together with an action $\alpha\colon Q \to \mathrm{Aut}(N)$ of the group G on N. The classifying spaces are rather special in that they are obtained from convex polytopes by suitably identifying facets. We shall say that such classifying spaces are *polytopal*. The action is also special in that any element $q \in Q$ acts on N by changing no more than the sign of generators. In the example the classifying space B_N is just 1-dimensional; for higher dimensional examples we should insist that each homomorphism $N \to N, n \mapsto {}^q n$ is induced by a cellular map $\tilde{\alpha}(q)\colon \tilde{B}_N \to \tilde{B}_N$ which, in each dimension, changes no more than the orientation of a face. We say that such an action α is a *polytopal action*. The arguments underlying our two proofs of the first assertion in Proposition 5 yield the following general result. We thank the referee for pointing out that this result is essentially a special case of the main result in the paper [4] by T. Brady.

Theorem 12 *[4] Let B_N and B_Q be polytopal classifying spaces of groups N and Q. Let $\alpha\colon Q \to \mathrm{Aut}(N)$ be a polytopal action and consider the semi-direct product $G = N \times_\alpha Q$. Then*

1. *$\tilde{B}_N \times \tilde{B}_Q$ is the universal covering space of a polytopal classifying space B_G for G.*
2. *the free $\mathbb{Z}G$-resolution $C_*(\tilde{B}_G)$ can be obtained as the total complex of a double complex with $\dim(B_N)$ rows and $\dim(B_Q)$ columns.*

This theorem yields a finite dimensional polytopal classifying space for quite a range of twisted Artin groups. Consider for instance the one-relator Artin groups $N = \langle x, y | xyx = yxy\rangle$, $Q = \langle s, t | sts = tst\rangle$ and action α of Q on N given by ${}^s x = x^{-1}$, ${}^t x = x^{-1}$, ${}^s y = y^{-1}$, ${}^t y = y^{-1}$. The semi-direct product $G = N \times_\alpha Q$ is then the twisted Artin group represented by the following graph.



Since the action is polytopal this twisted Artin group G admits a 4-dimensional classifying space obtained from the direct product of two hexagons by suitably identifying faces.

Theorem 12 implies the following special case of Theorem 14.

Proposition 13 *Let A be an n-generator directed, right-angled, twisted Artin group. If there are no ∞ edges in its Coxeter graph then A has a classifying space B_A with universal cover $\mathbb{R}^n$.*

Proof. Let $S = \{x_1, \ldots, x_n\}$ be the generating set for A. Since A is directed, it is possible to extend the Coxeter graph to a directed complete acyclic graph on S. The extended labelling induces a total order on S, say $x_1 < x_2 < \cdots < x_n$. We observe that A can be viewed as a semi-direct product $A = N \times_\alpha Q$ with N the infinite cyclic group and with Q the twisted Artin group corresponding to the complete subgraph of D obtained by removing vertex x_1 and its incident edges. By induction, Theorem 12 yields an n-dimensional classifying space for A with universal cover equal to $\mathbb{R}^n$. $\square$

The following is a generalisation of a result of Charney and Davis [9] for Artin groups.

Theorem 14 *If a right-angled twisted Artin group A is directed, then it admits a finite dimensional classifying space $K(A,1)$. The cells of $K(A,1)$ are indexed by the subsets of the generators of the associated Coxeter group W_A that generate finite subgroups. If no edge of the Coxeter graph of A is labelled by ∞ then A is a Bieberbach group.*

Proof. The theorem is known to hold for right-angled Artin groups [9, 8]. We use this known special case of the result to prove the general case. Let A be a directed right-angled twisted Artin group with generating set S. Let $\hat{A}$ denote the twisted Artin group on S whose Coxeter graph is obtained from that of A by replacing any label $m = \infty$ with the label $m = 2$. Thus, by Proposition 13, $\hat{A}$ is a Bieberbach group with $|S|$-dimensional classifying space $B_{\hat{A}}$. Let B_A be the subcomplex of $B_{\hat{A}}$ consisting of only those cells whose index $T \subset S$ generates a finite subgroup of the Coxeter group W_A.

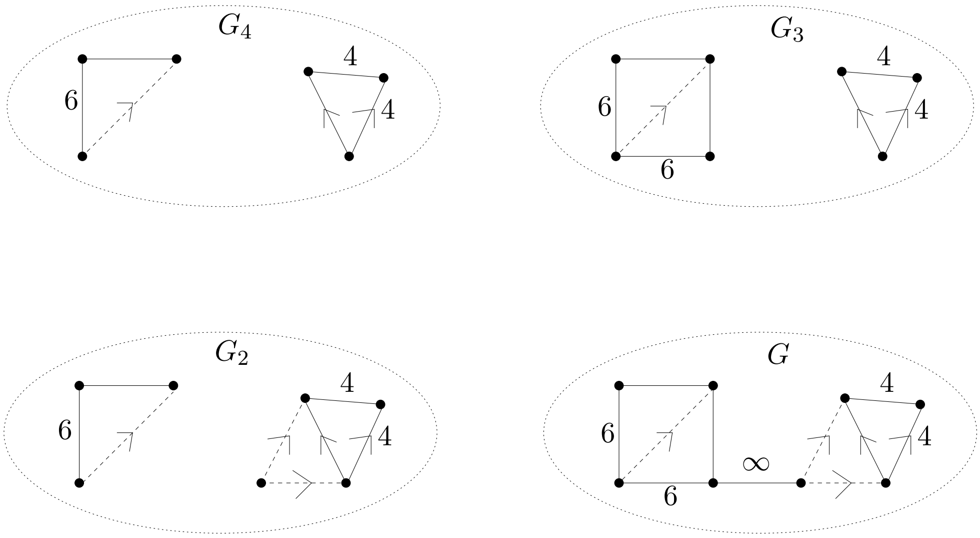
Let A' be the non-twisted right-angled Artin group obtained by removing all dotted edges from the Coxeter graph of A. Thus the graphs of A and A' have precisely the same ∞ edges. Define $B_{\hat{A}'}$ and $B_{A'}$ analogously to $B_{\hat{A}}$ and B_A. The result of Charney and Davis says that $B_{A'}$ is a classifying space for A'.

Consider the universal covering maps $\psi\colon \mathbb{R}^n \to B_{\hat{A}}$ and $\psi'\colon \mathbb{R}^n \to B_{\hat{A}'}$. Let X be the pre-image in $\mathbb{R}^n$ of the complex B_A, and let X' be the pre-image of the complex $B_{A'}$. We then have covering maps $\phi\colon X \to B_A$ and $\phi'\colon X' \to B_{A'}$. Let $\tilde{X}$ be the universal cover of X, and let $\tilde{X}'$ be the universal cover of X'. Then $\tilde{X}$ and $\tilde{X}'$ are the universal covers of B_A and $B_{A'}$. Since $B_{A'}$ is known to be a classifying space we have that $\tilde{X}'$ must be contractible. But $X = X'$ and hence $\tilde{X} = \tilde{X}'$. So $\tilde{X}$ is contractible. It follows that B_A is a classifying space for A as required. $\square$

8 The group defined by the relators in (1)

Finally we mention that Charney and Davis [8] give a finite-dimensional classifying space for the class of FC Artin groups which includes all right-angled Artin groups. An even larger class of Artin groups are shown to admit finite-dimensional classifying spaces in [17]. The proof in [17] is extremely short and can be directly applied to certain twisted Artin groups. In particular, it is a routine exercise to apply the method of the proof to the twisted Artin group G defined in the introduction to show that it admits a finite-dimensional classifying space. An outline of the application is as follows.

First consider the following four twisted Artin groups.



Using above results we can construct finite-dimensional classifying CW-spaces $B_{G_2}, B_{G_3}, B_{G_4}$ for the groups G_2, G_3, G_4 such that B_{G_4} is a subcomplex of B_{G_2} and B_{G_3}. In each of the three cases the Coxeter graph has two components and we take the classifying space B_{G_i} to be the direct product of the classifying spaces for the two components. The group G_4 is a direct product of a 2-dimensional 3-generator Artin group and a large type 3-generator twisted Artin group so we use Proposition 9 and Theorem 6 in the construction of B_{G_4}. The group G_3 is also a direct product of a 2-dimensional twisted Artin group (Proposition 9) with a 2-dimensional 3-generator twisted Artin group of large type (Theorem 6). The group G_2 is a direct product of a 2-dimensional Artin group with a semi-direct product of a large type twisted Artin group and infinite cyclic group; so we use Theorem 12 in the construction of B_{G_2}.

In this example it is easy to see that the inclusions $B_{G_4} \to B_{G_2}$ and $B_{G_4} \to B_{G_3}$ induce injective homomorphisms of fundamental groups. We construct the union $B_G = B_{G_2} \cup B_{G_3}$. Then van Kampen's theorem and a theorem of J.H.C. Whitehead imply that B_G is a classifying space for G. By construction, B_G is finite-dimensional.

The chain complex $C_*(B_G) \otimes_{\mathbb{Z}} \mathbb{Z}_2$ can be constructed using HAP [16] and used to compute the dimensions of the homology $H_n(G, \mathbb{Z}_2)$ given in the introduction.

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