HOMOTOPY 2-TYPES OF LOW ORDER

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Abstract. There is a well-known equivalence between the homotopy types of connected CW-spaces X with $\pi_n X=0$ for $n \neq 1,2$ and the quasi-isomorphism classes of crossed modules $\partial\colon M \to P$ [11]. When the homotopy groups $\pi_1 X$ and $\pi_2 X$ are finite one can represent the homotopy type of X by a crossed module in which M and P are finite groups. We define the *order* of such a crossed module to be $|\partial| = |M| \times |P|$, and the *order* of a quasi-isomorphism class of crossed modules to be the least order of any crossed module in the class. We then define the *order* of a homotopy 2-type X to be the order of the corresponding quasi-isomorphism class of crossed modules. In this paper we describe an implemented computer function that inputs a finite crossed module of reasonably small order and returns a quasi-isomorphic crossed module of least order. The function is used to enumerate all homotopy 2-types of order $m \leq 127$, $m \neq 32, 64, 81, 96$. Underlying the function is a catalogue of all isomorphism classes of crossed modules of order $m \leq 255$.

1. Introduction

An important resource for finite group theorists is the computer classification of all groups G of low order. This classification is available in the MAGAM [3] and GAP [7] computer systems and, for example, can be used to: (i) list representatives of all isomorphism classes of groups G of a given order m; (ii) identify the isomorphism class of a user-defined group G in terms of a pair (m,k) where m is the order of G and k is a catalogue number.

In this paper we build on work of Alp and Wensley [2] and develop the beginnings of an analogous resource for homotopy types of connected CW-spaces X with $\pi_n X = 0$ for $n \neq 1,2$. The homotopy type of X is called a *homotopy* 2*-type*. It is well-known that such a homotopy type can be modelled by a group homomorphism $\partial\colon M \to P$ and group action $(p,m) \mapsto {}^p m$ of P on M satisfying

(1) $\partial({}^p m) = p(\partial m)p^{-1}$
(2) ${}^{\partial m} m' = m m' m^{-1}$

for $p \in P, m, m' \in M$. Such a homomorphism and action constitute a *crossed module*. The model is such that $\pi_n X \cong \pi_n(\partial)$ for $n = 1,2$ where one defines $\pi_1(\partial) = P/\mathrm{im}\,\partial$ and $\pi_2(\partial) = \ker \partial$. A *morphism* of crossed modules $\phi_*\colon (\partial\colon M \to P) \to (\partial'\colon M' \to P')$ consists of two group homomorphisms $\phi_1\colon P \to P'$, $\phi_2\colon M \to M'$ satisfying $\partial'\phi_2(m) = \phi_1\partial(m)$ and $\phi_2({}^p m) = {}^{\phi_1 p}\, \phi_2(m)$ for $m, m' \in M, p \in P$. A morphism induces canonical homomorphisms $\pi_n(\phi_*)\colon \pi_n(\partial) \to \pi_n(\partial')$ for $n = 1,2$. The morphism ϕ_* is said to be an *isomorphism* if ϕ_n is

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an isomorphism for $n = 1, 2$. The morphism ϕ_* is said to be a *quasi-isomorphism* if $\pi_n(\phi_*)$ is an isomorphism for $n = 1, 2$. Two crossed modules ∂, ∂' are said to be *quasi-isomorphic* if there exists a sequence of quasi-isomorphisms $\partial \to \partial_1 \leftarrow \partial_2 \to \partial_3 \leftarrow \cdots \to \partial_n \leftarrow \partial'$ of arbitrary length n. We write $\partial \simeq \partial'$ to denote that ∂ is quasi-isomorphic to ∂'. Note that $\simeq$ is an equivalence relation on crossed modules; the corresponding equivalence classes are called *quasi-isomorphism classes*. We emphasize that two crossed module ∂, ∂' could be quasi-isomorphic without the existence of any quasi-isomorphism between ∂ and ∂'.

Mac Lane and Whitehead [11] showed that there is a one-one correspondence between homotopy 2-types and quasi-isomorphism classes of crossed modules. We define the *order* of a crossed module $\partial\colon M \to P$ to be the product $|\partial| = |M| \times |P|$ of the orders of the groups M, P. We define the *order* of a quasi-isomorphism class of crossed modules to be the least order of any crossed module in the class. We define the *order* of a homotopy 2-type X to be the order of the corresponding quasi-isomorphism class of crossed modules. A homotopy 2-type X can also be represented by the fundamental group $\pi_1 X$, the $\pi_1 X$-module $\pi_2 X$ and a cohomology class $\kappa \in H^3(\pi_1 X, \pi_2 X)$ known as the *Postnikov invariant*. The Postnikov invariant κ is the trivial cohomology class if and only if the homotopy 2-type can be represented by a crossed module $\partial\colon M \to P$ with $\partial = 0$. In this case we deem the homotopy 2-type, and also the quasi-isomorphism type, to be *trivial*.

In this paper we describe two computer functions, both of which have been implemented by the second author in the HAP package [5] for the computer algebra system GAP [7]. The first function lists representatives for all the quasi-isomorphism classes of crossed modules of a given order $m \leq 127$, $m \neq 32, 64, 81, 96$. The second function inputs a user-defined crossed module (of order possibly greater than 127) and tries to return numbers (m, k) that identify the least order m of any quasi-isomorphic crossed module and a catalogue number k that uniquely identifies the quasi-isomorphism class of the input; it certainly succeeds if the input is of order $\leq 127, \neq 32, 64, 81, 96$. We have used the implementation of these two functions, and related functions, to compile Table 1. The table uses the notation:

$$\begin{array}{lcl} Iso2(m) & = & \text{number of isomorphism classes of crossed modules of order } m. \\ QIso2(m) & = & \text{number of homotopy 2-types of order } m \\ & = & \text{number of quasi-isomorphism classes of order } m. \\ TQIso2(m) & = & \text{number of trivial homotopy 2-types of order } m. \end{array}$$

It is an easy exercise to see that $Iso2(p) = QIso2(p) = TQIso2(p) = 2$ for p a prime and so we omit prime values of m from the table. It is also easy to show that for primes $p < q$ we have $Iso2(pq) = QIso2(pq) = TQIso2(pq) = 6$ when p divides $q - 1$ and $Iso2(pq) = QIso2(pq) = TQIso2(pq) = 4$ when p does not divide $q-1$ and so these values of m are also omitted from the table. (To establish the formulae one uses that: the cyclic group of order p can act non-trivially on the cyclic group of order q precisely when p divides $q-1$; the only groups of order p or order pq with p not dividing $q-1$ are the cyclic groups; the only groups of order pq with p dividing $q-1$ are the cyclic group and one non-abelian semi-direct product of cyclic groups.)

m	1	4	8	9	12	16	18	20	24	25	27	28	30	32	36	40
$Iso2(m)$	1	6	18	6	20	62	22	20	73	6	18	18	20	251	78	72
$QIso2(m)$	1	5	14	5	18	43	19	18	61	5	14	16	20	A	63	60
$TQIso2(m)$	1	5	14	5	18	42	19	18	61	5	14	16	20	152	63	60

m	42	44	45	48	49	50	52	54	56	60	63	64	66	68	70	72
$Iso2(m)$	26	18	12	296	6	22	20	81	68	77	18	1276	20	20	20	325
$QIso2(m)$	26	16	10	224	5	19	18	65	56	73	16	B	20	18	20	251
$TQIso2(m)$	26	16	10	220	5	19	18	65	56	73	16	697	20	18	20	251

m	75	76	78	80	81	84	88	90	92	96	98	99	100	102	104
$Iso2(m)$	14	18	26	302	64	90	66	76	18	1446	22	12	87	20	72
$QIso2(m)$	12	16	26	230	C	84	54	66	16	D	19	10	71	20	60
$TQIso2(m)$	12	16	26	226	44	84	54	66	16	971	19	10	71	20	60

m	105	108	110	112	114	116	117	120	121	124	125	126	128
$Iso2(m)$	12	308	26	270	26	20	18	342	6	18	18	102	9120
$QIso2(m)$	12	238	26	202	26	18	16	302	5	16	14	92	?
$TQIso2(m)$	12	238	26	198	26	18	16	302	5	16	14	92	4668

$$158 \le A \le 171,\ 727 \le B \le 831,\ 45 \le C \le 46,\ 996 \le D \le 1052$$

TABLE 1

Perhaps not surprisingly, the table shows that most of the homotopy 2-types of low order have trivial Postnikov invariant. It shows that the smallest homotopy 2-type with non-trivial Postnikov invariant has order 16, and that there is just one non-trivial homotopy 2-type of this order. A straightforward computer analysis shows that this homotopy type is represented by the crossed module with $M = \langle x \mid x^4 = 1\rangle$, $P = \langle a \mid a^4 = 1\rangle$, ${}^a x = x^3$, $\partial(x) = a^2$. It is also represented by the crossed module $M = \langle x, y \mid x^2 = y^2 = [x, y] = 1\rangle$, $P = \langle a \mid a^4 = 1\rangle$, ${}^a x = xy$, ${}^a y = y$, $\partial(x) = a^2$, $\partial(y) = 1$. No other crossed module of order 16 represents the unique smallest homotopy 2-type with non-trivial Postnikov invariant. We remark that it has been observed previously that the second of the crossed modules representing this homotopy type corresponds to a non-trivial Postnikov invariant; see for instance the example of Section 7 in [8] and Example 12.7.12 in [4].

The values of $Iso2(m)$ for $m \le 63$ are available from the software [1] described in [2]. The values of $Iso2(m)$ for higher m are obtained from a function for listing non-isomorphic crossed modules of given order which was designed and implemented by the second author. Details are given in Section 3. We are grateful to Alexander Hulpke for providing a key step in the implementation of our algorithm for testing isomorphism of two crossed modules.

For each of the $Iso2(m)$ crossed modules of order m we apply a refinement of an algorithm in our previous paper [6] that attempts to find a smaller quasi-isomorphic crossed module. In this way we obtain an upper bound for $QIso2(m)$. Details are given in Section 4.

To prove that the upper bound equals $QIso2(m)$ we need a method for establishing that two crossed modules ∂, ∂' are not quasi-isomorphic. We do this by computing the quasi-isomorphism invariants $\pi_1(\partial)$, $\pi_2(\partial)$, $H^3(\pi_1(\partial), \pi_2(\partial))$ and $H_n(X, \mathbb{Z})$. The last invariant is the homology of the homotopy 2-type X represented by ∂ and is computed using our algorithm described in [6]. We use the group cohomology routines in our HAP package [5] to compute $H^3(\pi_1(\partial), \pi_2(\partial))$. Details are given in Section 5.

We begin by illustrating the functionality of our computer implementation in Section 2.

2. Computer implementation

It is well-known that the notion of a crossed module can be reformulated as a "group with compatible category structure". We use such a reformulation both for implementing algorithms and for checking correctness of algorithms. There are several variants of the reformulation and we opt to work with the following notion due to J-L. Loday [10].

A *cat^1-group* consists of a pair of group endomorphisms $s, t\colon G \to G$ satisfying $ts = s$, $st = t$ and $[\ker s, \ker t] = 1$. A *morphism* of cat^1-groups $\phi\colon (G, s, t) \to (G', s', t')$ consists of a group homomorphism $\phi\colon G \to G'$ satisfying $\phi s = s'\phi$ and $\phi t = t'\phi$. A cat^1-group gives rise to a crossed module by taking $M = \ker s$, $P = \mathrm{im}\, s$ and taking ∂ to be the restriction of t to $\ker s$. Conversely, a crossed module gives rise to a cat^1-group by using the action of P on M to form the semi-direct product $G = M \rtimes P$ and defining the endomorphisms $s, t\colon M \rtimes P \to M \rtimes P$ as $s(m, p) = (1, p)$, $t(m, p) = (1, (\partial m)p)$. It is observed in [10] that these two constructions provide an isomorphism between the category of crossed modules and the category of cat^1-groups. It is thus routine to translate notions of order, homotopy group, quasi-isomorphism and (trivial) quasi-isomorphism class of crossed modules to equivalent notions for cat^1-groups. We leave details to the reader and use these equivalent notions throughout the remainder of the paper.

Our first GAP session begins by setting G equal to the 500th group of order 2000 from the database of small groups. It then computes a list L of all possible non-isomorphic cat^1-group structures on G. The list L has length 16.

```
gap> G:=SmallGroup(2000,500);;

gap> L:=CatOneGroupsByGroup(G);;

gap> Length(L);
16
```

Our second GAP session involves the homomorphism $\alpha_H\colon H \to Aut(H), h \mapsto \iota_h$ from a group H to its automorphism group which sends $h \in H$ to the inner automorphism $\imath_h\colon H \to H, x \mapsto hxh^{-1}$. The homomorphism α_H is a crossed module with respect to the obvious action of $Aut(H)$ on H. The session begins by constructing the associated cat^1-group G_1 for H equal to the dihedral group of order 12. The second command in the session determines that the underlying group of G_1 is the 154th group of order 144 in the

small groups database, and that G_1 is endowed with the 8th cat^1-structure on this group. The session then constructs the cat^1-group G_2 associated to α_H for H the dihedral group of order 72 and computes $|G_2| = 31104$. The final command identifies the quasi-isomorphism class of G_2 to be the 55th quasi-isomorphism class of order 24.

```
gap> G1:=AutomorphismGroupAsCatOneGroup(DihedralGroup(12));;

gap> IdCatOneGroup(G1);
[ 144, 154, 8 ]

gap> G2:=AutomorphismGroupAsCatOneGroup(DihedralGroup(72));;

gap> Size(G2);
31104

gap> IdQuasiCatOneGroup(G2);
[ 24, 55 ]
```

Our third GAP session begins by constructing the cat^1-group corresponding to the 2nd homotopy 2-type X of order 30. It then uses the algorithm from [6] to compute $H_5(X, \mathbb{Z}) = \mathbb{Z}_{10}$.

```
gap> G:=SmallQuasiCatOneGroup(30,2);;

gap> Homology(G,5);
[ 10 ]
```

The implementation contains functions for converting a crossed module to a cat^1-group and vice-versa. Each of the above GAP sessions could thus equally well have been performed using equivalent crossed modules.

3. Enumeration of isomorphism classes

The GAP package [1] of Alp and Wensley provides a list of all non-isomorphic cat^1-groups and crossed modules of order ≤ 63. To handle larger examples the second author implemented a function which inputs a finite group G and outputs a list of all non-isomorphic cat^1-group structures (G, s, t). This implementation uses GAP's function `IdGroup(H)` for identifying certain subgroups $H \leq G$ by their order m and catalogue number k and thus works only in cases for which `IdGroup(H)` is implemented.

The algorithm begins by computing a list $\mathbb{L}$ of all normal subgroups N in G and a list $\mathbb{L}'$ of subgroups K in G representing all subgroup conjugacy classes. There are then two steps to the algorithm.

Step 1. For each $N \in \mathbb{L}$ we find all $K \in \mathbb{L}'$ satisfying

- K is isomorphic to G/N. (Here we just test if `IdGroup`(K) =`IdGroup`(G/N)).
- $|p(K)| = |G/N|$.

For each such pair N, K the quotient homomorphism $p\colon G \to G/N$ restricts to an isomorphism $p|_K\colon K \to G/N$. We form the inverse isomorphism $(p|_K)^{-1}\colon G/N \to K$ and set $\sigma = (p|_K)^{-1}p\colon G \to G$. By construction we have $\ker\sigma = N$, $\mathrm{im}\,\sigma = K$ and $\sigma\sigma = \sigma$. For each normal subgroup N we compute the list $\mathbb{L}_N$ of such homomorphisms σ.

Step 2. For each pair of normal subgroups N, M in G satisfying $[N, M] = 1$ we consider all $s \in \mathbb{L}_N$, $t \in \mathbb{L}_M$. If $\mathrm{im}\,s = \mathrm{im}\,t$ we add the data (G, s, t) to our list of cat^1-group structures on G.

In this manner, all possible cat^1-group structures on G are produced, though isomorphic copies may have been produced by the algorithm.

To test if two cat^1-group structures on a group G are isomorphic we need to access the automorphism group $Aut(G)$ of the group G. As this automorphism group can be large we follow a suggestion of Alexander Hulpke and use:

(i) the action ${}^fK = f(K)$ of $f \in Aut(G)$ on subgroups $K \leq G$;

(ii) the action ${}^fs(x) = fs(x)f^{-1}$ of $f \in Aut(G)$ on endomorphisms $s\colon G \to G$.

For each action we have adapted a GAP implementation of an orbit-stabilizer algorithm written by Alexander Hulpke and used it to compute the orbit of an element under the action and to compute the stabilizer subgroup of this element. A description of the orbit-stabilizer algorithm can be found in [9].

To test if two cat^1-group structures (G, s, t) and (G, s', t') are isomorphic we perform the following steps.

Step 1. We first use GAP's `IdGroup()` function to check that $\mathrm{im}\,s \cong \mathrm{im}\,s'$ and $\ker s \cong \ker s'$ and $\ker t \cong \ker t'$. If this check fails then the two cat^1-groups are not isomorphic and we return $false$.

Step 2. Otherwise we compute the orbit of $\ker s$ under the action of $Aut(G)$. If $\ker s'$ is not in this orbit then the two cat^1-groups are not isomorphic and we return $false$. Otherwise we can find an element $f \in Aut(G)$ such that $\ker s' = {}^f(\ker s)$. We then define $s'' = {}^{f^{-1}}s'$, $t'' = {}^{f^{-1}}t'$ to obtain a cat^1-group (G, s'', t'') which is isomorphic to (G, s', t') and which has the property that $\ker s'' = \ker s$. For ease of notation we redefine $s' := s''$, $t' := t''$. In other words, we replace (G, s', t') by an isomorphic cat^1-group staisfying $\ker s' = \ker s$.

Step 3. We compute the stabilizer subgroup $Stab(\ker s) \leq Aut(G)$ and the orbit of $\mathrm{im}\,s$ under the action of $Stab(\ker s)$. If $\mathrm{im}\,s'$ is not in this orbit then the two cat^1-groups are not isomorphic and we return $false$. Otherwise we can find an element $f \in Stab(\ker s)$ such that $\mathrm{im}\,s' = {}^f(\mathrm{im}\,s)$ and then replace (G, s', t') by an isomorphic cat^1-group satisfying $\mathrm{im}\,s' = \mathrm{im}\,s$ and $\ker s' = \ker s$.

Step 4. We compute the stabilizer subgroup $Stab(\mathrm{im}\,s) \leq Stab(\ker s)$ and the orbit of $\ker t$ under the action of $Stab(\mathrm{im}\,s)$. If $\ker t'$ is not in this orbit then the two cat^1-groups are not isomorphic and we return $false$. Otherwise we replace (G, s', t') by an isomorphic cat^1-group satisfying $\ker t' = \ker t$, $\mathrm{im}\,s' = \mathrm{im}\,s$ and $\ker s' = \ker s$.

Step 5. We compute the stabilizer $Stab(\ker t) \leq Stab(\mathrm{im}\,s)$ and the orbit of s under the action of $Stab(\ker t)$. If s' is not in this orbit the two cat^1-groups are not isomorphic and

we return $false$. Otherwise we replace (G, s', t') by an isomorphic cat^1-group satisfying $\ker t' = \ker t$, $s' = s$.

Step 6. We compute the stabilizer $Stab(s) \le Stab(\ker t)$ and the orbit of t under the action of $Stab(s)$. If t' is not in this orbit then the two cat^1-groups are not isomorphic and we return $false$. Otherwise we return true.

4. Construction of small quasi-isomorphic representatives

Given a cat^1-group (G, s, t) we attempt to find a smaller quasi-isomorphic cat^1-group using (a new implementation of) the following procedure which was described in [6].

Step 1: Given a finite cat^1-group (G, s, t) we search through the normal subgroups of $\ker s$ to find a largest normal subgroup K in $\ker s$ such that K is normal in G and $K \cap \ker t = 1$. We can then choose some generating set X_K for K and construct the group N generated by the set $X_K \cup \{t(x) : x \in X_K\}$. Then N is a normal subgroup of G for which the quotient homomorphism $G \to G/N$ is a quasi-isomorphism.
Step 2: Given a finite cat^1-group (G, s, t) we search through the subgroups of G to find a smallest cat^1-group $K \subset G$ such that the inclusion $K \hookrightarrow G$ is a quasi-isomorphism. Since we are dealing with finite groups we can test if this inclusion is a quasi-isomorphism simply by checking if the induced homomorphisms $\pi_n K \to \pi_n G$ are surjections and $|\pi_n K| = |\pi_n G|$ for $n = 1, 2$.
Step 3: We repeatedly apply (i) followed by (ii) until no more size reduction is achieved.

5. Establishing distinct quasi-isomorphism classes

A table of all isomorphism types of cat^1-groups of order at most 255 has been computed and stored in [5]. This table underlies the function `IdCatOneGroup(G)` which returns a triple (m, k_1, k_2) for any cat^1-group G of order $m \le 255$; the integer k_1 is the number of the underlying group of G in the small groups database; the integer k_2 is the number of the cat^1-group structure on G in our table of small cat^1-groups.

The table of isomorphism types of cat^1-groups immediately yields the upper bound $Iso2(m) \ge QIso2(m)$ on the number of quasi-isomorphism types of cat^1-groups. By applying the procedure of Section 4 to each isomorphism type G of order m in the table, and then discarding G if the procedure succeeds in finding a smaller cat^1-group quasi-isomorphic to G, we obtain a list L of representatives of quasi-isomorphism types of cat^1-groups of order m. It could be that some pair $G, G' \in L$ are quasi-isomorphic. For each $G \in L$ we have computed the following quasi-isomorphism invariants:

i) the small groups database identifier for $\pi_1(G)$;

ii) the small groups database identifier for $\pi_2(G)$;

iii) the small groups database identifier for the semi-direct product $\pi_2(G) \rtimes \pi_1(G)$ involving the action of the first homotopy group on the second homotopy group;

iv) the abelian invariants of the integral homology group $H_n(X, \mathbb{Z})$ for $n \le 5$ where X is the homotopy 2-type represented by G. The homology algorithm from [6] was used for this.

If for every pair of non-isomorphic cat^1-groups $G, G' \in L$ at least one of the invariants (i)-(iv) yields distinct values we conclude that our list L contains precisely one representative for each quasi-isomorphism class of order m.

In cases where invariants (i)-(iv) are identical for two cat^1-groups $G, G' \in L$ the order of the cohomology group $H^3(\pi_1(G), \pi_2(G))$ was computed using group cohomology functions in [5]. By Mac Lane and Whitehead's result [11], the order of this cohomology group provides an upper bound on the number of quasi-isomorphism types with given fundamental group $\pi_1(G)$ and given second homotopy group $\pi_2(G)$. In some cases this upper bound is sufficient to conclude that G and G' are quasi-isomorphic.

Using the above strategy a non-redundant list of representatives of all quasi-isomorphism types of order m has been computed and recorded in the software [5] for all $m \le 127$ excluding $m = 32, 64, 81, 96$. This record is used to implement the function `IdQuasiCatOneGroup(G)` which inputs a cat^1-group G and tries to return the the pair of integers (m, k) with m the order of the smallest cat^1-group in the quasi-isomorphism class of G and k a number uniquely identifying this smallest representative. The function first attempts to find a quasi-isomorphism representative of G of order $\le 127, \ne 32, 64, 81, 96$; if it succeeds it then uses the stored record to produce m and k.

For $m = 32, 64, 81, 96$ the above strategy produces the bounds $158 \le QIso2(32) \le 171$, $727 \le Iso2(64) \le 831$, $45 \le Iso2(81) \le 46$ and $996 \le Iso2(96) \le 1052$.

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