Basic Perturbation Lemma and effective homology: application to the computation of homology of 2-types

Ana Romero

Universidad de La Rioja (Spain)

(Joint work with J. Rubio and F. Sergeraert)

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Introduction

The Basic Perturbation Lemma was discovered by Shih Weishu in 1960, and the abstract modern form was given by Ronnie Brown in 1964 (after some unpublished results by M. Barrat).

We have used it combined with the effective homology method, in order to determine:

- Homology of cones, bicomplexes, twisted Cartesian products, loop spaces, classifying spaces...
- Homotopy groups of spaces by means of Whitehead and Postnikov towers.
- Homology of digital images by means of Discrete Vector Fields.
- Spectral sequences associated with filtered complexes (including Serre and Eilenberg-Moore spectral sequences).
- Persistent homology.
- Koszul homology.
- Bousfield-Kan spectral sequence for computing homotopy groups of spaces.
- Homology of groups.
- Homology of 2-types.
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- Homology of 2-types.
Effective homology

Definition

A reduction $\rho$ between two chain complexes $C^\ast$ and $D^\ast$ (denoted by $\rho: C^\ast \Rightarrow D^\ast$) is a triple $\rho = (f, g, h)$ satisfying the following relations:

1) $fg = \text{Id}_{D^\ast}$;
2) $d_{C^\ast}h + hd_{C^\ast} = \text{Id}_{C^\ast} - gf$;
3) $fh = 0$; $hg = 0$; $hh = 0$.

If $C^\ast \Rightarrow D^\ast$, then $C^\ast \cong D^\ast \oplus A^\ast$, with $A^\ast$ acyclic, which implies that $H_n(C^\ast) \cong H_n(D^\ast)$ for all $n$. 
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Effective homology

Definition

A (strong chain) equivalence \( \varepsilon \) between \( C^* \) and \( D^* \), \( \varepsilon : C^* \leftrightarrow D^* \), is a triple \( \varepsilon = (B^*, \rho, \rho') \) where \( B^* \) is a chain complex, \( \rho : B^* \rightarrow C^* \) and \( \rho' : B^* \rightarrow D^* \).

Definition

An object with effective homology is a quadruple \( (X, C^*(X), EC^*, \varepsilon) \) where \( EC^* \) is an effective chain complex and \( \varepsilon : C^*(X) \leftrightarrow EC^* \). This implies that \( H_n(X) \cong H_n(EC^*) \) for all \( n \).
**Effective homology**

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A *(strong chain) equivalence* \( \varepsilon \) between \( C_* \) and \( D_* \), \( \varepsilon : C_* \iff D_* \), is a triple \( \varepsilon = (B_*, \rho, \rho') \) where \( B_* \) is a chain complex, \( \rho : B_* \Rightarrow C_* \) and \( \rho' : B_* \Rightarrow D_* \).

\[ \begin{array}{ccc}
C_* & \xrightarrow{\rho} & B_* \\
& \searrow & \nearrow \\
& 14 & 42 \\
& 10 & 30 \\
& \nearrow & \searrow \\
D_* & \xrightarrow{\rho'} & B_* \\
& \swarrow & \nwarrow \\
& 21 & 15 \\
& \nwarrow & \swarrow \\
\end{array} \]
Effective homology

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A **(strong chain) equivalence** $\varepsilon$ between $C_\ast$ and $D_\ast$, $\varepsilon : C_\ast \leftrightarrow D_\ast$, is a triple $\varepsilon = (B_\ast, \rho, \rho')$ where $B_\ast$ is a chain complex, $\rho : B_\ast \Rightarrow C_\ast$ and $\rho' : B_\ast \Rightarrow D_\ast$.

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\downarrow & & \downarrow \\
14 & \xrightarrow{10} & 21 \\
30 & \xrightarrow{42} & 15
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Definition

An **object with effective homology** is a quadruple $(X, C_\ast(X), EC_\ast, \varepsilon)$ where $EC_\ast$ is an effective chain complex and $\varepsilon : C_\ast(X) \leftrightarrow EC_\ast$. This implies that $H_n(X) \cong H_n(EC_\ast) \quad \forall \ n$. 

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Basic Perturbation Lemma, effective homology and homology of 2-types

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Effective homology

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**Definition**

An object with effective homology is a quadruple $(X, C_*(X), EC_*, \varepsilon)$ where $EC_*$ is an effective chain complex and $\varepsilon : C_*(X) \iff EC_*$. 

This implies that $H_n(X) \cong H_n(EC_*)$ for all $n$. 

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HPT Workshop
Effective homology

Meta-theorem

Let $X_1, \ldots, X_k$ be a collection of objects with effective homology and $\Phi$ be a reasonable construction process:

$$\Phi : (X_1, \ldots, X_k) \rightarrow X.$$ 

Then there exists a version with effective homology $\Phi_{EH}: ((X_1, C(X_1), EC_1, \varepsilon_1), \ldots, (X_k, C(X_k), EC_k, \varepsilon_k)) \rightarrow (X, C(X), EC, \varepsilon).$

The process is perfectly stable and can be again used with $X$ for further calculations.

Examples:

twisted Cartesian products, loop spaces, suspensions, simplicial Abelian groups generated by simplicial sets, . . . .
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The process is perfectly stable and can be again used with $X$ for further calculations.

**Examples:** twisted Cartesian products, loop spaces, suspensions, simplicial Abelian groups generated by simplicial sets, . . . .
The Kenzo system

The Kenzo system uses the notion of object with effective homology to compute homology groups of some complicated spaces. If the complex is effective, then its homology groups can be determined by means of diagonalization algorithms on matrices. Otherwise, the program uses the effective homology.

Example:

\[ X = \Omega(\Omega(\Omega(P_\infty \mathbb{R}/P_3 \mathbb{R}) \cup D_4) \cup D_2) \]

\[ H_5(X) = \mathbb{Z}_2^2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{16} \]

\[ H_6(X) = \mathbb{Z}_5^2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_3 \]

\[ H_7(X) = \mathbb{Z}_{113}^2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{3}^8 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}_{32} \]
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\[
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\]

\[
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\]

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Example:

\[
X = \Omega(\Omega(\Omega(\mathbb{P}_\infty \mathbb{R} / \mathbb{P}_3 \mathbb{R})) \cup 4 \mathbb{D}^4) \cup 2 \mathbb{D}^2)
\]

\[
H_5(X) = \mathbb{Z}_2^3 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{16}
\]

\[
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\]

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H_5(X) = \mathbb{Z}_2^{23} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{16}
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Perturbation theorems

Definition

Let $(C^*, d)$ be a chain complex. A perturbation $\delta: C^* \to C^{*-1}$ is an operator of degree $-1$ satisfying $(d + \delta) \circ (d + \delta) = 0$. This produces a new perturbed chain complex $(C^*, d + \delta)$.

Let $\rho = (f, g, h)$ be a reduction $(C^*, d_C) \downarrow \downarrow f \to (D^*, d_D) \leftarrow g \leftarrow \leftarrow h$. What happens if we perturb $d_C$ or $d_D$?
Definition

Let \((C_\ast, d)\) be a chain complex. A perturbation \(\delta : C_\ast \rightarrow C_{\ast - 1}\) is an operator of degree \(-1\) satisfying \((d + \delta) \circ (d + \delta) = 0\).
Definition

Let \((C_*, d)\) be a chain complex. A *perturbation* \(\delta : C_* \rightarrow C_{*-1}\) is an operator of degree \(-1\) satisfying \((d + \delta) \circ (d + \delta) = 0\).

This produces a new *perturbed* chain complex \((C_*, d + \delta)\).
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Let \((C_*, d)\) be a chain complex. A \textit{perturbation} \(\delta : C_* \to C_{* - 1}\) is an operator of degree \(-1\) satisfying \((d + \delta) \circ (d + \delta) = 0\).

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Let \(\rho = (f, g, h)\) be a reduction

\[
\begin{align*}
(C_*, d_C) & \xleftarrow{g} (D_*, d_D) \\
\xrightarrow{f} & \\
\end{align*}
\]
**Definition**

Let \((C_*, d)\) be a chain complex. A **perturbation** \(\delta : C_* \to C_{*-1}\) is an operator of degree \(-1\) satisfying \((d + \delta) \circ (d + \delta) = 0\).

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What happens if we perturb \(d_C\) or \(d_D\)?
Perturbation theorems

Theorem (Trivial Perturbation Lemma, TPL)

Let \( \rho = (f, g, h) : C^* \Rightarrow D^* \) be a reduction, and \( \delta_D \) a perturbation of \( d_D \).

Then we have a new reduction:

\[
(C^*, d_C + \delta_C) \downarrow f \rightarrow (D^*, d_D + \delta_D) \leftarrow g
\]

where \( \delta_C = g \circ \delta_D \circ f \).
Theorem (Trivial Perturbation Lemma, TPL)

Let \( \rho = (f, g, h) : C_\ast \Rightarrow D_\ast \) be a reduction, and \( \delta_D \) a perturbation of \( d_D \).
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Let $\rho = (f, g, h) : C_* \Rightarrow D_*$ be a reduction, and $\delta_D$ a perturbation of $d_D$.

Then we have a new reduction: $(C_*, d_C + \delta_C) \rightleftarrows (D_*, d_D + \delta_D)$

where $\delta_C = g \circ \delta_D \circ f$. 

Diagram:

```
  h
  \downarrow
  f

(C_*, d_C + \delta_C) \rightarrow (D_*, d_D + \delta_D)
```

where $g$.
Theorem (Basic Perturbation Lemma, BPL)

Let \( \rho = (f, g, h) : C^* \Rightarrow D^* \) be a reduction, and \( \delta_C \) a perturbation of \( C \) such that the composition \( h \circ \delta_C \) is pointwise nilpotent.

Then we have a new reduction:

\[
(C^*, d_C + \delta_C) \xrightarrow{f'} \xrightarrow{g'} (D^*, d_D + \delta_D)
\]

where \( \delta_D = f \circ \delta_C \circ \phi \circ g = f \circ \psi \circ \delta_C \circ g; \)
\( f' = f \circ \psi = f \circ (\text{Id}_{C^*} - \delta_C \circ \phi \circ h) \);
\( g' = \phi \circ g; \)
\( h' = \phi \circ h = h \circ \psi \);

with the operators \( \phi \) and \( \psi \) defined by

\[
\phi = \sum_{i=0}^{\infty} (-1)^i (h \circ \delta_C)^i, \quad \psi = \sum_{i=0}^{\infty} (-1)^i (\delta_C \circ h)^i = \text{Id}_{C^*} - \delta_C \circ \phi \circ h
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Theorem (Basic Perturbation Lemma, BPL)

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Then we have a new reduction:

\[
(C_*, d_C + \delta_C) \downarrow = (D_*, d_D + \delta_D)\]

where

\[
\delta_D = f \circ \delta_C \circ \phi \circ g \]
\[
f' = f \circ \psi = f \circ (id_C - \delta_C \circ \phi \circ h)\]
\[
g' = \phi \circ g\]
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\[
\begin{align*}
(C_*, d_C + \delta_C) & \xrightarrow{h'} (D_*, d_D + \delta_D) \\
& \xleftarrow{f'}
\end{align*}
\]

with the operators \( \phi \) and \( \psi \) defined by

\[
\phi = \sum_{i=0}^{\infty} (-1)^i (h \circ \delta_C)_i \quad \text{and} \quad \psi = \sum_{i=0}^{\infty} (-1)^i (\delta_C \circ h)_i = \text{Id}_{C_*} - \delta_C \circ \phi \circ h.
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\begin{array}{ccc}
C_* & \xleftarrow{\delta_C} & D_* \\
\downarrow{h'} & & \downarrow{f'} \\
C_* & \xleftarrow{\delta_C} & D_* \\
\end{array}
\]

where

\[
\delta_D = f \circ \delta_C \circ \phi \circ g = f \circ \psi \circ \delta_C \circ g;
\]

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f' = f \circ \psi = f \circ (\text{Id}_{C_*} - \delta_C \circ \phi \circ h);
\]

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g' = \phi \circ g;
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h' = \phi \circ h = h \circ \psi;
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\]

\[
\psi = \sum_{i=0}^{\infty} (-1)^i (\delta_C \circ h)^i = \text{Id}_{C_*} - \delta_C \circ \phi \circ h
\]
Algebraic cone construction

**Definition**

Let $\Phi : (\mathbb{C}^*, d_{\mathbb{C}}) \to (\mathbb{D}^*, d_{\mathbb{D}})$ be a chain complex morphism. The **cone** of $\Phi$, Cone$(\Phi) ≡ (\mathbb{A}^*, d_{\mathbb{A}})$, is a chain complex given by

$A_n = \mathbb{C}_n \oplus \mathbb{D}_{n+1}$,

with differential map $d_{\mathbb{A}}(c, d) = (d_{\mathbb{C}}(c), \Phi(c) - d_{\mathbb{D}}(d))$.
**Definition**

Let $\Phi : (C_*, d_C) \to (D_*, d_D)$ be a chain complex morphism. The *Cone* of $\Phi$, $\text{Cone}(\Phi)_* \equiv (A_*, d_A)$, is a chain complex given by $A_n = C_n \oplus D_{n+1}$, with differential map $d_A(c, d) = (d_C(c), \Phi(c) - d_D(d))$. 
Definition

Let \( \Phi : (C_*, d_C) \rightarrow (D_*, d_D) \) be a chain complex morphism. The Cone of \( \Phi \), \( \text{Cone}(\Phi)_* \equiv (A_*, d_A) \), is a chain complex given by \( A_n = C_n \oplus D_{n+1} \), with differential map \( d_A(c, d) = (d_C(c), \Phi(c) - d_D(d)) \).
Algebraic cone construction
A general algorithm can be produced:

- **Input**: $\Phi : C_* \to D_*$ and effective homologies for $C_*$ and $D_*$. 
- **Output**: An effective homology for $A_* = \text{Cone}(\Phi)$. 

**Theorem**
Theorem

A general algorithm can be produced:

- **Input:** $\Phi : C_* \rightarrow D_*$ and effective homologies for $C_*$ and $D_*$.  
- **Output:** An effective homology for $A_* = \text{Cone}(\Phi)$.

Proof:
Theorem

A general algorithm can be produced:

- **Input:** $\Phi : C_* \rightarrow D_*$ and effective homologies for $C_*$ and $D_*$.  
- **Output:** An effective homology for $A_* = \text{Cone}(\Phi)$. 

**Proof:**

$$
\begin{array}{ccc}
C_* & \xrightarrow{h} & D_* \\
\downarrow f & & \downarrow h' \\
EC_* & \xrightarrow{g} & ED_*
\end{array}
$$
A general algorithm can be produced:

- **Input:** \( \Phi : C_* \rightarrow D_* \) and effective homologies for \( C_* \) and \( D_* \).
- **Output:** An effective homology for \( A_* = \text{Cone}(\Phi) \).

**Proof:**

1. Particular case \( \Phi = 0 \) (direct sum).
Theorem

A general algorithm can be produced:

- **Input:** $\Phi : C_* \to D_*$ and effective homologies for $C_*$ and $D_*$. 
- **Output:** An effective homology for $A_* = \text{Cone}(\Phi)$.

Proof:

1. Particular case $\Phi = 0$ (direct sum).

\[
\begin{align*}
\begin{bmatrix}
  0 & 0 & 0 & h' \\
  0 & 0 & 0 & -h' \\
  f & g & f' & g' \\
  d_C & 0 & d_{EC} & 0 \\
  0 & -d_D & 0 & -d_{ED}
\end{bmatrix}
\end{align*}
\]
Theorem

A general algorithm can be produced:

- **Input:** $\Phi : C_* \to D_*$ and effective homologies for $C_*$ and $D_*$.  
- **Output:** An effective homology for $A_* = \text{Cone}(\Phi)$.  

Proof:

1. Particular case $\Phi = 0$ (direct sum).
2. We install $\Phi$.  

The reduction is not valid.

\[
\begin{bmatrix}
    d_C & 0 & 0 \\
    \phi & -d_D & 0 \\
\end{bmatrix}
\begin{bmatrix}
    d_{EC} & 0 & 0 \\
    0 & -d_{ED} & 0 \\
\end{bmatrix}
\begin{bmatrix}
    f & 0 & 0 \\
    0 & f' & 0 \\
\end{bmatrix}
\begin{bmatrix}
    g & 0 & 0 \\
    0 & g' & 0 \\
\end{bmatrix}
\begin{bmatrix}
    h & 0 & 0 \\
    0 & -h' & 0 \\
\end{bmatrix}
\]

$D_A \quad D_{A'} \quad F \quad G \quad H$
Algebraic cone construction

Theorem

A general algorithm can be produced:

- Input: $\Phi : C_* \rightarrow D_*$ and effective homologies for $C_*$ and $D_*$. 
- Output: An effective homology for $A_* = \text{Cone}(\Phi)$.

Proof:

1. Particular case $\Phi = 0$ (direct sum).
2. We install $\Phi$.
3. We apply the BPL.

\[
\begin{pmatrix}
  d_C & 0 \\
  \Phi & -d_D \\
\end{pmatrix}
\begin{pmatrix}
  d_{EC} & 0 \\
  f' \Phi g & -d_{ED} \\
\end{pmatrix}
\begin{pmatrix}
  f & 0 \\
  f' \Phi h & f' \\
\end{pmatrix}
\begin{pmatrix}
  g & 0 \\
  -h' \Phi g & g' \\
\end{pmatrix}
\begin{pmatrix}
  h & 0 \\
  h' \Phi h & -h' \\
\end{pmatrix}
\]

$DA$  $DA'$  $F$  $G$  $H$
A bicomplex $C^\bullet,\bullet$ is a bigraded free $\mathbb{Z}$-module $C^p,q$ for $p, q \in \mathbb{Z}$ provided with morphisms $d^\prime_{p,q}: C_{p,q} \rightarrow C_{p-1,q}$ and $d^\prime\prime_{p,q}: C_{p,q} \rightarrow C_{p,q-1}$ satisfying $d^\prime_{p-1,q} \circ d^\prime_{p,q} = 0$, $d^\prime\prime_{p,q-1} \circ d^\prime\prime_{p,q} = 0$, and $d^\prime_{p,q-1} \circ d^\prime\prime_{p,q} + d^\prime\prime_{p-1,q} \circ d^\prime_{p,q} = 0$.

The total (chain) complex $T^\bullet = T^\bullet(C^\bullet,\bullet) = (T_n,d_n)$ for $n \in \mathbb{Z}$ is the chain complex given by $T_n = \bigoplus_{p+q = n} C_{p,q}$ and differential map $d_n(x) = d^\prime_{p,q}(x) + d^\prime\prime_{p,q}(x)$ for $x \in C_{p,q}$.
A bicomplex $C_{*,*}$ is a bigraded free $\mathbb{Z}$-module $C_{*,*} = \{C_{p,q}\}_{p,q \in \mathbb{Z}}$ provided with morphisms $d'_p,q : C_{p,q} \rightarrow C_{p-1,q}$ and $d''_p,q : C_{p,q} \rightarrow C_{p,q-1}$ satisfying $d'_{p-1,q} \circ d'_p,q = 0$, $d''_{p,q-1} \circ d''_p,q = 0$, and $d'_{p,q-1} \circ d''_p,q + d''_{p-1,q} \circ d'_p,q = 0$. 
A bicomplex $C_{*,*}$ is a bigraded free $\mathbb{Z}$-module $C_{*,*} = \{ C_{p,q} \}_{p,q \in \mathbb{Z}}$ provided with morphisms $d'_p,q : C_{p,q} \to C_{p-1,q}$ and $d''_{p,q} : C_{p,q} \to C_{p,q-1}$ satisfying $d'_{p-1,q} \circ d'_p,q = 0$, $d''_{p,q-1} \circ d''_{p,q} = 0$, and $d'_{p,q-1} \circ d''_{p,q} + d''_{p-1,q} \circ d'_p,q = 0$.

The total (chain) complex $T_* = T_*(C_{*,*}) = (T_n, d_n)_{n \in \mathbb{Z}}$ is the chain complex given by $T_n = \bigoplus_{p+q=n} C_{p,q}$ and differential map $d_n(x) = d'_p,q(x) + d''_{p,q}(x)$ for $x \in C_{p,q}$. 

\hspace{1cm}
A **bicomplex** $C_{*, *}$ is a bigraded free $\mathbb{Z}$-module $C_{*, *} = \{ C_{p, q} \}_{p, q \in \mathbb{Z}}$ provided with morphisms $d'_p,q : C_{p, q} \to C_{p-1, q}$ and $d''_p,q : C_{p, q} \to C_{p, q-1}$ satisfying $d'_{p-1,q} \circ d'_p,q = 0$, $d''_{p-1,q} \circ d''_p,q = 0$, and $d'_p,q \circ d''_p,q + d''_{p-1,q} \circ d'_p,q = 0$.

The **total (chain) complex** $T_* = T_*(C_{*, *}) = (T_n, d_n)_{n \in \mathbb{Z}}$ is the chain complex given by $T_n = \bigoplus_{p+q = n} C_{p, q}$ and differential map $d_n(x) = d'_p,q(x) + d''_p,q(x)$ for $x \in C_{p, q}$. 
Theorem

A general algorithm can be produced:

Input: A bounded bicomplex $C^*$ and effective homologies of each column.

Output: An effective homology for $C^*$.

Proof:
Theorem

A general algorithm can be produced:

- Input: A bounded bicomplex $C_\ast$ and effective homologies of each column.
- Output: An effective homology for $C_\ast$. 

Proof:
A general algorithm can be produced:

- **Input:** A bounded bicomplex $C_*$ and effective homologies of each column.
- **Output:** An effective homology for $C_*$. 

**Proof:**

1. We consider only the vertical arrows.

\[
\begin{array}{cccc}
C_{0,3} & C_{1,3} & C_{2,3} & C_{3,3} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
C_{0,2} & C_{1,2} & C_{2,2} & C_{3,2} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
C_{0,1} & C_{1,1} & C_{2,1} & C_{3,1} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
C_{0,0} & C_{1,0} & C_{2,0} & C_{3,0} \\
\end{array}
\]

\[
\begin{array}{cccc}
EC_{0} & EC_{1} & EC_{2} & EC_{3} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
EC_{0} & EC_{1} & EC_{2} & EC_{3} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
EC_{0} & EC_{1} & EC_{2} & EC_{3} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
EC_{0} & EC_{1} & EC_{2} & EC_{3} \\
\end{array}
\]
Theorem

A general algorithm can be produced:

- **Input**: A bounded bicomplex $C_*$ and effective homologies of each column.
- **Output**: An effective homology for $C_*$.

Proof:

1. We consider only the vertical arrows.

\[\begin{array}{cccccc}
C_0,0 & C_1,0 & C_2,0 & C_3,0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
EC_0 & EC_1 & EC_2 & EC_3 \\
\end{array}\]

\[\begin{array}{cccccc}
C_0,1 & C_1,1 & C_2,1 & C_3,1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
EC_1 & EC_1 & EC_2 & EC_3 \\
\end{array}\]

\[\begin{array}{cccccc}
C_0,2 & C_1,2 & C_2,2 & C_3,2 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
EC_2 & EC_2 & EC_2 & EC_3 \\
\end{array}\]

\[\begin{array}{cccccc}
C_0,3 & C_1,3 & C_2,3 & C_3,3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
EC_3 & EC_3 & EC_3 & EC_3 \\
\end{array}\]
A general algorithm can be produced:

- **Input**: A bounded bicomplex $C_\ast$ and effective homologies of each column.
- **Output**: An effective homology for $C_\ast$.

**Proof:**

1. We consider only the vertical arrows.

2. We *perturb* by adding the horizontal maps.
A general algorithm can be produced:

- **Input:** A bounded bicomplex $C_*$ and effective homologies of each column.
- **Output:** An effective homology for $C_*$.

**Proof:**

1. We consider only the vertical arrows.

2. We perturb by adding the horizontal maps.

3. We apply the BPL.
A general algorithm can be produced:

- **Input:** A bounded bicomplex $C_\ast$ and effective homologies of each column.
- **Output:** An effective homology for $C_\ast$.

**Proof:**

1. We consider only the vertical arrows.
2. We perturb by adding the horizontal maps.
3. We apply the BPL.
Theorem (Twisted Eilenberg-Zilber)

Given two simplicial sets $G$ and $B$ and a twisting operator $\tau : B \to G$, it is possible to construct a reduction $\rho = (f, g, h) : C^\ast(G \times \tau B) \Rightarrow C^\ast(G) \otimes t C^\ast(B)$ where $C^\ast(G) \otimes t C^\ast(B)$ is a chain complex with the same underlying graded module as the tensor product $C^\ast(G) \otimes C^\ast(B)$, but the differential is modified to take account of the twisting operator $\tau$. 

Proof: BPL.
Theorem (Eilenberg-Zilber)

Given two simplicial sets $G$ and $B$, there exists a reduction

$$\rho = (f, g, h) : C_*(G \times B) \Rightarrow C_*(G) \otimes C_*(B)$$
Theorem (Eilenberg-Zilber)

Given two simplicial sets $G$ and $B$, there exists a reduction

$$\rho = (f, g, h) : C_\ast(G \times B) \Rightarrow C_\ast(G) \otimes C_\ast(B)$$

Theorem (Twisted Eilenberg-Zilber)

Given two simplicial sets $G$ and $B$ and a twisting operator $\tau : B \to G$, it is possible to construct a reduction

$$\rho = (f, g, h) : C_\ast(G \times \tau B) \Rightarrow C_\ast(G) \otimes_t C_\ast(B)$$

where $C_\ast(G) \otimes_t C_\ast(B)$ is a chain complex with the same underlying graded module as the tensor product $C_\ast(G) \otimes C_\ast(B)$, but the differential is modified to take account of the twisting operator $\tau$. 
Twisted Eilenberg-Zilber Theorem

**Theorem (Eilenberg-Zilber)**

Given two simplicial sets $G$ and $B$, there exists a reduction

$$
\rho = (f, g, h) : C_*(G \times B) \Rightarrow C_*(G) \otimes C_*(B)
$$

**Theorem (Twisted Eilenberg-Zilber)**

Given two simplicial sets $G$ and $B$ and a twisting operator $\tau : B \rightarrow G$, it is possible to construct a reduction

$$
\rho = (f, g, h) : C_*(G \times_\tau B) \Rightarrow C_*(G) \otimes_\tau C_*(B)
$$

where $C_*(G) \otimes_\tau C_*(B)$ is a chain complex with the same underlying graded module as the tensor product $C_*(G) \otimes C_*(B)$, but the differential is modified to take account of the twisting operator $\tau$.

**Proof:** BPL.
Theorem

A general algorithm can be produced:

Input: two simplicial sets $G$ and $B$ (where $B$ is $1$-reduced), a twisting operator $\tau : B \to G$, and effective homologies for $G$ and $B$.

Output: An effective homology for $E = G \times \tau B$.

Proof:

It is constructed as the composition of two equivalences:

$$C^\ast(G \times \tau B) \xrightarrow{\rho_1} \xrightarrow{\rho_2} \xrightarrow{\rho_3} C^\ast(G) \otimes tC^\ast(B) \xrightarrow{\rho_2} \xrightarrow{\rho_3} C^\ast(G \times \tau B)$$

where $\rho_2$ and $\rho_3$ are obtained by applying the TPL and the BPL respectively.
Theorem

A general algorithm can be produced:
- **Input**: two simplicial sets $G$ and $B$ (where $B$ is 1-reduced), a twisting operator $\tau : B \to G$, and effective homologies for $G$ and $B$.
- **Output**: An effective homology for $E = G \times_\tau B$. 

Proof:
It is constructed as the composition of two equivalences:

\[
\begin{align*}
\rho_1 &\downarrow & \rho_2 &\downarrow & \rho_3 \\
\downarrow & & \downarrow & & \downarrow \\
C^* (G \times_\tau B) &\cong & C^* (G) \otimes t^* C^* (B) &\cong & C^* (E) \otimes t^* C^* (B)
\end{align*}
\]

where $\rho_2$ and $\rho_3$ are obtained by applying the TPL and the BPL respectively.
Effective homology of twisted Cartesian products

**Theorem**

A general algorithm can be produced:

- **Input:** two simplicial sets $G$ and $B$ (where $B$ is 1-reduced), a twisting operator $\tau : B \rightarrow G$, and effective homologies for $G$ and $B$.
- **Output:** An effective homology for $E = G \times_{\tau} B$.

**Proof:** It is constructed as the composition of two equivalences:

$$
\begin{align*}
C_\ast(G \times_{\tau} B) & \xrightarrow{\text{Id}} C_\ast(G \times_{\tau} B) \\
C_\ast(G) \otimes_t C_\ast(B) & \xrightarrow{\rho_1} C_\ast(G) \otimes_t C_\ast(B) \\
DG_\ast \otimes_t DB_\ast & \xrightarrow{\rho_2} DG_\ast \otimes_t DB_\ast \\
EG_\ast \otimes_t EB_\ast & \xrightarrow{\rho_3} EG_\ast \otimes_t EB_\ast
\end{align*}
$$

where $\rho_2$ and $\rho_3$ are obtained by applying the TPL and the BPL respectively.
Theorem

A general algorithm can be produced:

- **Input:** two simplicial sets $G$ and $B$ (where $B$ is 1-reduced) and a twisting operator $\tau : B \to G$, and effective homologies for $B$ and $E = G \times_\tau B$.
- **Output:** An effective homology for $G$.
Effective homology of the fiber of a fibration

**Theorem**
A general algorithm can be produced:

- **Input:** two simplicial sets $G$ and $B$ (where $B$ is 1-reduced) and a twisting operator $\tau : B \to G$, and effective homologies for $B$ and $E = G \times_{\tau} B$.
- **Output:** An effective homology for $G$.

**Proof:** It is constructed as the composition of two equivalences:

\[
\begin{align*}
\text{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z}) & \xrightarrow{\text{Id}} C_*(G) \\
C_*(G) & \xrightarrow{\text{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z})} \text{Cobar}^{\text{DB}_*}(DE_*, \mathbb{Z}) \\
& \xrightarrow{\text{Cobar}^{\text{EB}_*}(EE_*, \mathbb{Z})} \text{Cobar}^{\text{EB}_*}(EE_*, \mathbb{Z})
\end{align*}
\]
Effective homology of the fiber of a fibration

**Theorem**

A general algorithm can be produced:

- **Input:** two simplicial sets $G$ and $B$ (where $B$ is 1-reduced) and a twisting operator $\tau : B \to G$, and effective homologies for $B$ and $E = G \times_\tau B$.

- **Output:** An effective homology for $G$.

**Proof:** It is constructed as the composition of two equivalences:

\[
\begin{align*}
\text{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z}) & \cong DB_* \left( DE_*, \mathbb{Z} \right) \\
C_*(G) & \cong \text{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z}) & \cong EB_* \left( EE_*, \mathbb{Z} \right)
\end{align*}
\]

In particular, it can be applied for computing the effective homology of a loop space $\Omega(X)$, which is the fiber of a fibration $\Omega(X) \hookrightarrow \Omega(X) \times_\tau X \to X$ where the total space $E = \Omega(X) \times_\tau X$ is contractible, such that a reduction $C_*\left(\Omega(X) \times_\tau X\right) \Rightarrow \mathbb{Z}$ can be built.
Discrete Morse theory

Definition
Let $C^* = (C^p, d_p)_{p \in \mathbb{Z}}$ be a free chain complex with distinguished $\mathbb{Z}$-basis $\beta_p \subset C^p$. A discrete vector field $V$ on $C^*$ is a collection of pairs $V = \{ (\sigma_i; \tau_i) \}_{i \in I}$ satisfying the conditions:

1. Every $\sigma_i$ is some element of $\beta_p$, in which case $\tau_i \in \beta_p + 1$.
2. The degree $p$ depends on $i$ and in general is not constant.
3. Every component $\sigma_i$ is a regular face of the corresponding $\tau_i$.
4. Each generator (cell) of $C^*$ appears at most one time in $V$.

Definition
A $V$-path of degree $p$ and length $m$ is a sequence $\pi = (\sigma_i k, \tau_i k)_{0 \leq k < m}$ satisfying:

1. Every pair $(\sigma_i k, \tau_i k)$ is a component of $V$ and $\tau_i k$ is a $p$-cell.
2. For every $0 < k < m$, the component $\sigma_i k$ is a face of $\tau_i k - 1$, non necessarily regular, but different from $\sigma_i k - 1$.
Discrete Morse theory

### Definition

Let $C_* = (C_p, d_p)_{p \in \mathbb{Z}}$ a free chain complex with distinguished $\mathbb{Z}$-basis $\beta_p \subset C_p$. A *discrete vector field* $V$ on $C_*$ is a collection of pairs $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ satisfying the conditions:

- Every $\sigma_i$ is some element of $\beta_p$, in which case $\tau_i \in \beta_{p+1}$. The degree $p$ depends on $i$ and in general is not constant.
- Every component $\sigma_i$ is a *regular face* of the corresponding $\tau_i$.
- Each generator (*cell*) of $C_*$ appears at most one time in $V$. 

**Ana Romero**

Basic Perturbation Lemma, effective homology and homology of 2-types  

HPT Workshop 17 / 29
Definition

Let $C_\ast = (C_p, d_p)_{p \in \mathbb{Z}}$ a free chain complex with distinguished $\mathbb{Z}$-basis $\beta_p \subset C_p$. A discrete vector field $V$ on $C_\ast$ is a collection of pairs $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ satisfying the conditions:

- Every $\sigma_i$ is some element of $\beta_p$, in which case $\tau_i \in \beta_{p+1}$. The degree $p$ depends on $i$ and in general is not constant.
- Every component $\sigma_i$ is a regular face of the corresponding $\tau_i$.
- Each generator (cell) of $C_\ast$ appears at most one time in $V$.

Definition

A $V$-path of degree $p$ and length $m$ is a sequence $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0 \leq k < m}$ satisfying:

- Every pair $((\sigma_{i_k}, \tau_{i_k}))$ is a component of $V$ and $\tau_{i_k}$ is a $p$-cell.
- For every $0 < k < m$, the component $\sigma_{i_k}$ is a face of $\tau_{i_{k-1}}$, non necessarily regular, but different from $\sigma_{i_{k-1}}$. 
Discrete Morse theory

Definition
A discrete vector field $V$ is admissible if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every $V$-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

Definition
A cell $\sigma$ which does not appear in a discrete vector field $V$ is called a critical cell.

Theorem
Let $C^* = (C_p, d_p)$ $p \in \mathbb{Z}$ be a free chain complex and $V = \{(\sigma_i; \tau_i)\} i \in I$ be an admissible discrete vector field on $C^*$. Then the vector field $V$ defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rightarrow (C^c_p, d'_p)$ where $C^c_p = \mathbb{Z}[\beta^c_p]$ is the free $\mathbb{Z}$-module generated by the critical $p$-cells.

Proof:
Uses BPL.
Discrete Morse theory

**Definition**

A discrete vector field $V$ is *admissible* if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every $V$-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$. 

**Theorem**

Let $C^* = (C_p, d_p)$ $\in \mathbb{Z}$ be a free chain complex and $V = \{ (\sigma_i; \tau_i) \}_{i \in I}$ be an admissible discrete vector field on $C^*$. Then the vector field $V$ defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rightarrow (C_c p, d' p)$ where $C_c p = \mathbb{Z}[\beta_c p]$ is the free $\mathbb{Z}$-module generated by the critical $p$-cells.

**Proof:** Uses BPL.
Discrete Morse theory

**Definition**

A discrete vector field $V$ is *admissible* if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every $V$-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

**Definition**

A cell $\sigma$ which does not appear in a discrete vector field $V$ is called a *critical cell*. 
**Definition**

A discrete vector field $V$ is *admissible* if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every $V$-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

**Definition**

A cell $\sigma$ which does not appear in a discrete vector field $V$ is called a *critical cell*.

**Theorem**

Let $C_* = (C_p, d_p)_{p \in \mathbb{Z}}$ be a free chain complex and $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ be an admissible discrete vector field on $C_*$. Then the vector field $V$ defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rightarrow (C^c_p, d'_p)$ where $C^c_p = \mathbb{Z}[\beta^c_p]$ is the free $\mathbb{Z}$-module generated by the critical $p$-cells.
**Definition**

A discrete vector field $V$ is *admissible* if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every $V$-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

**Definition**

A cell $\sigma$ which does not appear in a discrete vector field $V$ is called a **critical cell**.

**Theorem**

Let $C_* = (C_p, d_p)_{p \in \mathbb{Z}}$ be a free chain complex and $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ be an admissible discrete vector field on $C_*$. Then the vector field $V$ defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rightarrow (C^c_p, d'_p)$ where $C^c_p = \mathbb{Z}[\beta^c_p]$ is the free $\mathbb{Z}$-module generated by the critical $p$-cells.

**Proof:** Uses BPL.
16 vertices
24 edges
8 squares
Discrete Morse theory and digital images

16 vertices
24 edges
8 squares
Discrete Morse theory and digital images

- 16 vertices
- 24 edges
- 8 squares

- 1 vertex
- 1 edge
Other applications of BPL and effective homology

Homotopy groups of spaces by means of Whitehead and Postnikov towers.
Spectral sequences of filtered complexes.
Persistent homology.
Koszul homology.
Bousfield-Kan spectral sequence.
Homology of groups.
Homology of 2-types.
Homotopy groups of spaces by means of Whitehead and Postnikov towers.
Other applications of BPL and effective homology

- Homotopy groups of spaces by means of Whitehead and Postnikov towers.

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Other applications of BPL and effective homology

- Homotopy groups of spaces by means of Whitehead and Postnikov towers.
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- Homology of 2-types.
Homology of groups

Definition
A 2-type \( X \) is a topological space with \( \pi_i(X) = 0 \) for \( i \neq 1, 2 \).

Definition
A resolution \( F^* \) for a group \( G \) is an acyclic chain complex of \( \mathbb{Z}G \)-modules:

\[
\cdots \rightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} F_{-1} \rightarrow \mathbb{Z} \rightarrow 0
\]

A chain complex of Abelian groups is obtained:

\[
\mathbb{Z} \otimes \mathbb{Z}G F^*
\]

Theorem
Let \( G \) be a group and \( F^*, F'^* \) two free resolutions of \( G \). Then

\[
H_n(\mathbb{Z} \otimes \mathbb{Z}G F^*) \cong H_n(\mathbb{Z} \otimes \mathbb{Z}G F'^*)
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for all \( n \in \mathbb{N} \).
Homology of groups

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Homology of groups

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Homology of groups

Definition
Given a group \( G \), the homology groups \( H_n(G) \) are defined as
\[
H_n(G) = \text{Hom}(\mathbb{Z} \otimes \mathbb{Z}G, F*)
\]
where \( F* \) is any free resolution for \( G \).

Drawback: for \( n > 1 \), \( K(G, 1)_n = G_n \).

For some particular cases, small (or minimal) resolutions can be directly constructed.

For instance, let \( G = \mathbb{C}_m \) with generator \( t \). The resolution
\[
F* \rightarrow t^{-1} \rightarrow \mathbb{Z}G \rightarrow t \rightarrow \mathbb{Z} \rightarrow 0
\]
produces
\[
H_i(G) = \begin{cases} 
\mathbb{Z} & \text{if } i = 0 \\
\mathbb{Z}/m\mathbb{Z} & \text{if } i \text{ is odd} \\
0 & \text{if } i \text{ is even}, i > 0
\end{cases}
\]
Homology of groups

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Given a group $G$, the *homology groups* $H_n(G)$ are defined as

$$H_n(G) = H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*), \quad n \in \mathbb{N},$$

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One can always consider the bar resolution $B_*$, which satisfies

$$\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \equiv C_*(K(G,1)).$$

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Homology of groups

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Basic Perturbation Lemma, effective homology and homology of 2-types
Homology of groups

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Basic Perturbation Lemma, effective homology and homology of 2-types

HPT Workshop 22 / 29
Homology of groups

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Basic Perturbation Lemma, effective homology and homology of 2-types
HPT Workshop
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\end{cases}$$
Algorithm computing the effective homology of a group

Given $G$ a group, $F^\ast$ a (small) free $\mathbb{Z}_G$-resolution with a contracting homotopy $h^n: F^n \to F^{n+1}$.

Goal: an equivalence $C^\ast(K(G,1)) \iff E^\ast$ where $E^\ast$ is an effective chain complex.

We consider the bar resolution $B^\ast = \text{Bar}^\ast(G)$ for $G$ with contracting homotopy $h'$.

It is well known that there exists a morphism of chain complexes of $\mathbb{Z}_G$-modules $f: B^\ast \to F^\ast$ which is a homotopy equivalence.

An algorithm has been designed constructing the explicit expressions of $f$ and the corresponding maps $g, h, k$. 
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Algorithm computing the effective homology of a group

Applying the functor $\mathbb{Z} \otimes \mathbb{Z}^G$ we obtain an equivalence of chain complexes (of $\mathbb{Z}$-modules):

$$
\begin{array}{c}
\mathbb{Z} \otimes \mathbb{Z}^G \mathcal{B}^\ast \\
\downarrow \downarrow f \\
\mathbb{Z} \otimes \mathbb{Z}^G \mathcal{F}^\ast \\
\leftarrow \leftarrow g \\
\downarrow \downarrow k
\end{array}
$$

In order to obtain a strong chain equivalence we make use of the mapping cylinder construction.

$$
\begin{array}{c}
\mathbb{Z} \otimes \mathbb{Z}^G \mathcal{B}^\ast \\
\leftarrow \leftarrow \rho' \\
\Cylinder(f) \ast \rho \\
\Rightarrow \Rightarrow \\
\mathbb{Z} \otimes \mathbb{Z}^G \mathcal{F}^\ast
\end{array}
$$

Finally we observe that the left chain complex $\mathbb{Z} \otimes \mathbb{Z}^G \mathcal{B}^\ast$ is equal to $\mathcal{C}^\ast(\mathcal{K}(\mathbb{G}, 1))$. Moreover, if the initial resolution $\mathcal{F}^\ast$ is of finite type (and small), then the right chain complex $\mathbb{Z} \otimes \mathbb{Z}^G \mathcal{F}^\ast \equiv \mathcal{E}^\ast$ is effective.
Applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ we obtain an equivalence of chain complexes (of $\mathbb{Z}$-modules):

$$
\begin{array}{ccc}
\mathbb{Z} \otimes_{\mathbb{Z}G} B_* & \xrightarrow{f} & \mathbb{Z} \otimes_{\mathbb{Z}G} F_* \\
\mathbb{Z} \otimes_{\mathbb{Z}G} B_* & \leftarrow \quad \text{Cylinder}(f) & \quad \leftarrow \mathbb{Z} \otimes_{\mathbb{Z}G} F_*
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Finally we observe that the left chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ is equal to $C^*(K(G,1))$. Moreover, if the initial resolution $F_*$ is of finite type (and small), then the right chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} F_*$ is effective.
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\xleftarrow{\rho} & \xleftarrow{\rho'} & \xrightarrow{\rho}
\end{array}
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Algorithm computing the effective homology of a group

Theorem

A general algorithm can be produced:

**Input:** a group $G$ and a free resolution $F^{\ast}$ of finite type with contracting homotopy.

**Output:** the effective homology of $K(G,1)$, that is, a (strong chain) equivalence $C^{\ast}(K(G,1)) \iff E^{\ast}$ where $E^{\ast}$ is an effective chain complex.

Implemented in Common Lisp, enhancing the Kenzo system. It allows to compute homology of groups and, what is more important, to use the space $K(G,1)$ in other constructions allowing new computations.

Ana Romero

Basic Perturbation Lemma, effective homology and homology of 2-types

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Homology of 2-types

Let $G = C_3$, $A = \mathbb{Z}/3\mathbb{Z}$ with trivial $G$-action, and $f \in H^3(G, A) = \mathbb{Z}/3\mathbb{Z}$ a non-trivial cohomology class. This corresponds to a 2-type with $\pi_1 = G$ and $\pi_2 = A$, which can be seen as $X = K(A, 2) \times_f K(G, 1)$.

Finite (small) resolutions are known for $G$ and $A$. The spaces $K(A, 1)$ and $K(G, 1)$ have effective homology thanks to the previous algorithm. From the effective homology of $K(A, 1)$, the effective homology of the classifying space $W K(A, 1) = K(A, 2)$ is built. From the effective homologies of $K(A, 2)$ and $K(G, 1)$, we construct the effective homology of $X = K(A, 2) \times_f K(G, 1)$. 

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Basic Perturbation Lemma, effective homology and homology of 2-types
HPT Workshop 26 / 29
Let $G = C_3$, $A = \mathbb{Z}/3\mathbb{Z}$ with trivial $G$-action, and $[f] \in H^3(G, A) = \mathbb{Z}/3\mathbb{Z}$ a non-trivial cohomology class.
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- Finite (small) resolutions are known for $G$ and $A$.
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- From the effective homologies of $K(A, 2)$ and $K(G, 1)$, we construct the effective homology of $X = K(A, 2) \times_f K(G, 1)$.  

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Basic Perturbation Lemma, effective homology and homology of 2-types  
HPT Workshop 26 / 29
Homology of 2-types

Kenzo computes the homology groups of the 2-type:

```
> (setf K-C3-1 (K-Cm-n 3 1))
> (setf chml-clss (chml-clss K-C3-1 3))
> (setf tau (zp-whitehead 3 K-C3-1 chml-clss))
> (setf x (fibration-total tau))
> (efhm x)
```

Homology in dimension 5:
Component $\mathbb{Z}/3\mathbb{Z}$
---done---
Kenzo computes the homology groups of the 2-type:
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```lisp
> (setf K-C3-1 (K-Cm-n 3 1))
[K261 Abelian-Simplicial-Group]
> (setf chml-clss (chml-clss K-C3-1 3))
[K308 Cohomology-Class on K288 of degree 3]
> (setf tau (zp-whitehead 3 K-C3-1 chml-clss))
[K323 Fibration K261 -> K309]
> (setf x (fibration-total tau))
[K329 Kan-Simplicial-Set]
> (efhm x)
[K541 Homotopy-Equivalence K329 <= K531 => K527]
> (homology x 5)
Homology in dimension 5:
Component Z/3Z
---done---
```
Conclusions

The Basic Perturbation Lemma is not basic. Combined with the effective homology method, it can be used for computing homology and homotopy groups of different spaces and other constructions of Algebraic Topology such as spectral sequences, persistent homology, homology of groups...

In particular, it allows us to determine homology of 2-types.
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Combined with the effective homology method, it can be used for computing homology and homotopy groups of different spaces and other constructions of Algebraic Topology such as spectral sequences, persistent homology, homology of groups. . .

In particular, it allows us to determine homology of 2-types.
Basic Perturbation Lemma and effective homology: application to the computation of homology of 2-types

Ana Romero

Universidad de La Rioja (Spain)

(Joint work with J. Rubio and F. Sergeraert)

Homological Perturbation Theory Workshop
Galway, December 2014