Robust control theory as a basic application of the homological perturbation lemma

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Homological Perturbation Lemma
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Outline of the talk

1. Control theory in a nutshell
2. Stabilities and stabilization problems
3. Robust control theory ($H^\infty$ control theory)
4. Robust control theory as a trivial application of the homological perturbation lemma
Control theory in a nutshell

1. **Modeling problems**: Find a correct mathematical model for a real system coming from physics, engineering sciences, . . .

2. **Analysis problems**: Analyze the properties of the system

3. **Synthesis problems**: Construct a feedback controller which stabilizes and optimizes the performances of the closed-loop system, study the robustness issues, . . .
Single input - single output (SISO) control systems

- **Ordinary differential equation:**
  \[ \dot{z}(t) = z(t) + u(t), \quad z(0) = 0, \quad y(t) := z(t). \]

- **Differential time-delay equation:** \( h > 0 \)
  \[
  \begin{aligned}
  \dot{z}(t) &= z(t) + u(t), \quad x(0) = 0, \\
  y(t) &= \begin{cases} 
  0, & 0 \leq t < h, \\
  z(t - h), & t \geq h.
  \end{cases}
  \end{aligned}
  \]

- **Partial differential equation:** (wave equation)
  \[
  \begin{aligned}
  \frac{\partial^2 z}{\partial t^2}(x, t) - a^2 \frac{\partial^2 z}{\partial x^2}(x, t) &= 0, \\
  z(x, 0) &= 0, \quad \frac{\partial z}{\partial t}(x, 0) = 0, \\
  z(0, t) &= u(t), \quad z(L, t) = 0, \\
  y(t) &= z(l, t), \quad l \text{ is a fixed point in } [0, L].
  \end{aligned}
  \]
• Partial differential equation: (an electric transmission line)

\[
\begin{align*}
\frac{\partial V}{\partial x}(x, t) + L \frac{\partial I}{\partial t}(x, t) + R I(x, t) &= 0, \\
\frac{\partial I}{\partial x}(x, t) + C \frac{\partial V}{\partial t}(x, t) + G V(x, t) &= 0, \\
V(x, 0) &= 0, \quad I(x, 0) = 0, \\
V(0, t) &= u(t), \quad \lim_{x \to +\infty} V(x, t) = 0, \\
y_1(t) &:= V(L, t), \quad y_2(t) := I(L, t),
\end{align*}
\]
**Laplace transform**

- **Definition:** The Laplace transform of \( f \in L_1(\mathbb{R}_+) \) is defined by:
  \[
  \hat{f}(s) := \int_0^{+\infty} e^{-st} f(t) \, dt, \quad \forall s \in \mathbb{C}_\alpha := \{ s \in \mathbb{C} \mid \text{Re } s > \alpha \}.
  \]

1. \( \hat{f} \) is analytic and bounded in \( \mathbb{C}_\alpha \).
2. If \( (f \ast g)(t) := \int_0^t f(t - \tau) g(\tau) \, d\tau, \ t \geq 0 \), then:
   \[
   \hat{f} \ast \hat{g} = \hat{f} \hat{g}.
   \]
3. If \( g(t) := \begin{cases} f(t - h), & t \geq h, \\ 0, & 0 < t < h, \end{cases} \), then \( \hat{g}(s) = e^{-hs} \hat{f}(s) \).
4. If \( f \) is \( n \) times differentiable for \( t > 0 \) and \( f^{(1)}, \ldots, f^{(n)} \) are Laplace transformable, then:
   \[
   \hat{f}^{(n)}(s) = s^n \hat{f}(s) - \sum_{i=0}^{n-1} s^{n-i-1} f^{(i)}(0_+).
   \]
Transfer functions and modes

- **Ordinary differential equation:**
  \[ \dot{z}(t) = z(t) + u(t), \quad z(0) = 0 \quad y(t) = z(t), \quad \Rightarrow \quad \hat{y}(s) = \frac{1}{(s - 1)} \hat{u}(s). \]

- **Differential time-delay equation:** \( h > 0 \)
  \[
  \begin{align*}
  \dot{z}(t) &= z(t) + u(t), \quad x(0) = 0, \\
y(t) &= \begin{cases} 
  0, & 0 \leq t < h, \\
z(t - h), & t \geq h,
\end{cases}
\end{align*}
\]
  \[ \Rightarrow \quad \hat{y}(s) = \frac{e^{-hs}}{(s - 1)} \hat{u}(s). \]

- **Partial differential equation:**
  \[
  \begin{align*}
  \frac{\partial^2 z}{\partial t^2}(x, t) - a^2 \frac{\partial^2 z}{\partial x^2}(x, t) &= 0, \\
z(x, 0) &= 0, \quad \frac{\partial z}{\partial t}(x, 0) = 0, \quad \Rightarrow \quad \hat{y}(s) = \frac{\left( e^{-\frac{l}{a}s} - e^{-\frac{(2L-l)}{a}s} \right)}{\left( 1 - e^{-\frac{2a}{L}s} \right)} \hat{u}(s).
\end{align*}
\]
Stabilities

- Let us define the right half plane \( \mathbb{C}_+ := \{ s \in \mathbb{C} \mid \text{Re} \, s > 0 \} \).

- The Hardy algebra \( H^\infty(\mathbb{C}_+) \) (Banach algebra) is defined by:
  \[
  H^\infty(\mathbb{C}_+) := \{ \text{holomorphic fcts in } \mathbb{C}_+ \mid \| f \|_\infty = \sup_{s \in \mathbb{C}_+} |f(s)| < +\infty \}.
  \]

- The Hardy algebra \( H^\infty(\mathbb{C}_+) \) is the algebra of transfer functions of \( L^2(\mathbb{R}_+) - L^2(\mathbb{R}_+) \)-stable shift-invariant \( \infty \)-dimensional systems.

- \( RH^\infty := \mathbb{R}(s) \cap H^\infty(\mathbb{C}_+) \)
  \[
  = \left\{ \frac{n}{d} \mid 0 \neq d, n \in \mathbb{R}[s], \deg n \leq \deg d, d(s_*) = 0 \Rightarrow \text{Re} \, s_* < 0 \right\}
  \]
  is the algebra of exponentially-stable finite-dimensional plants.
Stabilities

- $L^1(\mathbb{R}_+) = \{ f : [0, +\infty] \to \mathbb{R} | \| f \|_1 = \int_0^{+\infty} |f(t)| \, dt < +\infty \}$,

- $l^1(\mathbb{Z}_+) = \{ a : \mathbb{Z}_+ = \{0, 1, \ldots \} \to \mathbb{R} | \| (a_i)_{i \in \mathbb{Z}_+} \|_1 = \sum_{i=0}^{+\infty} |a_i| < +\infty \}$.

- The Wiener algebra $\mathcal{A}$ is defined by:

$$\mathcal{A} := \{ f = g + \sum_{i=0}^{+\infty} a_i \delta(t-h_i) | \ g \in L^1(\mathbb{R}_+), \ (a_i)_{i \in \mathbb{Z}_+} \in l^1(\mathbb{Z}_+), \ \ 0 = h_0 \leq h_1 \leq h_2 \leq \cdots \}.$$  

- $\mathcal{A}$ is a Banach algebra w.r.t. $\| f \|_{\mathcal{A}} := \| g \|_1 + \| (a_i)_{i \in \mathbb{Z}_+} \|_1$.

- $\hat{\mathcal{A}} := \{ \hat{f} | \ f \in \mathcal{A} \}$, $\| \hat{f} \|_{\hat{\mathcal{A}}} = \| f \|_{\mathcal{A}}$.

- $\mathcal{A}$ is the algebra of $L^\infty(\mathbb{R}_+) - L^\infty(\mathbb{R}_+)$-stable shift-invariant $\infty$-dimensional systems.
The fractional representation approach

- (Zames) The set of transfer functions has the structure of an algebra (parallel $+$, series $\circ$, proportional feedback $\cdot$ by $\mathbb{R}$).

- (Vidyasagar) Let $A$ be an algebra of stable transfer functions with a structure of an integral domain ($ab = 0 \Rightarrow a = 0 \lor b = 0$).

$$Q(A) := \{ \frac{n}{d} \mid 0 \neq d, n \in A \}$$ represents the class of systems.

$$p \in A \iff p \text{ is stable}, \quad p \in K \setminus A \iff p \text{ is unstable}$$

- (Zames) The algebra $A$ should be a normed algebra so that the errors & the approximations of the real plant by the mathematical model can be considered

$$\text{(e.g., Banach algebra: } \| ab \|_A \leq \| a \|_A \cdot \| b \|_A, \quad \| 1 \|_A = 1\text{).}$$
Stability as a membership problem

Set of stable systems $A$

Set of all the systems $K := Q(A)$

unstable systems
Examples

• $\mathcal{R}H_{\infty}$ (algebra of exponentially-stable finite-dimensional plants):

$$p = \frac{1}{s - 1} = \left(\frac{1}{s+1}\right) \left(\frac{s-1}{s+1}\right), \quad \frac{1}{s+1}, \quad \frac{s-1}{s+1} \in \mathcal{R}H_{\infty} \Rightarrow p \in Q(\mathcal{R}H_{\infty}).$$

• $\hat{A}$ (algebra of BIBO-stable $\infty$-dimensional plants):

$$p = \frac{e^{-hs}}{s - 1} = \left(\frac{e^{-hs}}{s+1}\right) \left(\frac{s-1}{s+1}\right), \quad \frac{e^{-hs}}{s+1}, \quad \frac{s-1}{s+1} \in \hat{A} \Rightarrow p \in Q(\hat{A}).$$

• $H^\infty(\mathbb{C}_+)$ (algebra of $L^2(\mathbb{R}_+)$-stable $\infty$-dimensional plants):

$$p = \frac{(1 + e^{-2s})}{(1 - e^{-2s})} \in Q(H^\infty(\mathbb{C}_+)) : 1 + e^{-2s}, \ 1 - e^{-2s} \in H^\infty(\mathbb{C}_+).$$
Internal stabilization

- Let $A$ be an algebra of stable transfer function, $K := Q(A)$.
- Let $p \in K$ be a plant and $c \in K$ a controller.

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix} = \begin{pmatrix}
  1 & -p \\
  -c & 1
\end{pmatrix} \begin{pmatrix}
  e_1 \\
  e_2
\end{pmatrix}, \quad \begin{cases}
  y_1 = e_2 - u_2, \\
  y_2 = e_1 - u_1.
\end{cases}
\]

- **Definition:** $p \in K$ is stabilizable iff $\exists c \in K$:

\[
\begin{pmatrix}
  1 & -p \\
  -c & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
  \frac{1}{1-pc} & \frac{p}{1-pc} \\
  \frac{c}{1-pc} & \frac{1}{1-pc}
\end{pmatrix} \in A^{2 \times 2}.
\]
Example

- Example: $A = RH_\infty$, $K := Q(A) = \mathbb{R}(s)$.

\[
\begin{align*}
    p &= \frac{s}{s-1}, \\
    c &= -\frac{(s-1)}{(s+1)},
\end{align*}
\]

$\Rightarrow$ $c$ does not stabilize $p$ because:

\[
\frac{s(s+1)}{(2s+1)(s-1)} \notin RH_\infty \quad \text{(pole at 1 $\in \mathbb{C}_+$)}.
\]

- Example: $A = RH_\infty$, $K := Q(A) = \mathbb{R}(s)$.

\[
\begin{align*}
    p &= \frac{s}{s-1}, \\
    c &= 2,
\end{align*}
\]

$\Rightarrow$ $c$ stabilizes the plant $p$. 
Robust stabilizability

- Let $A$ be a Banach algebra of stable transfer functions.

**Definition:** Let $c \in K := Q(A)$ be a stabilizing controller of $p \in K$. Then, $c$ robustly stabilizes $p$ if there exists $\epsilon > 0$ such that $c$ internally stabilizes one of the family of plants:

1. **Additive perturbations:**
   
   $$B_1(p, \delta) := \{ p + \delta \mid \forall \delta \in A, \| \delta \|_A < \epsilon \}.$$

2. **Multiplicative perturbations:**
   
   $$B_3(p, \delta) := \{ p(1 + \delta) \mid \forall \delta \in A, \| \delta \|_A < \epsilon \}.$$

3. **Inverse additive perturbations:**
   
   $$B_2(p, \delta) := \{ p/(1 + \delta p) \mid \forall \delta \in A, \| \delta \|_A < \epsilon \}.$$

4. **Inverse multiplicative perturbations:**
   
   $$B_4(p, \delta) := \{ p/(1 + \delta) \mid \forall \delta \in A, \| \delta \|_A < \epsilon \}.$$
A fractional ideal approach

• Let $A$ be an integral domain, $K := Q(A) = \left\{ \frac{n}{d} \mid 0 \neq d, \ n \in A \right\}$.

• **Definition:** A **fractional ideal** $J$ of $A$ is an $A$-submodule of $K$,

\[
\forall \ a_1, a_2 \in A, \ \forall \ j_1, j_2 \in J : \ a_1 j_1 + a_2 j_2 \in J,
\]

such that $\exists \ d^* \in A \setminus \{0\}$ satisfying:

\[
J \subseteq \left( \frac{1}{d^*} \right) := \left\{ \frac{a}{d^*} \in K \mid a \in A \right\}.
\]

• **Example:** Let $p \in K$. Then,

\[
J = (1, \ p) := A + Ap = \{ \alpha + \beta p \mid \alpha, \beta \in A \}
\]

is a fractional ideal of $A$ since $\exists \ 0 \neq d, \ n \in A : \ p = \frac{n}{d}$

\[
\Rightarrow \ J = \left\{ \alpha + \beta \frac{n}{d} = \frac{\alpha d + \beta n}{d} \mid \alpha, \beta \in A \right\} \subseteq \left( \frac{1}{d} \right).
\]
Theory of fractional ideals

- **Proposition:** Let $\mathcal{F}(A)$ be the set of non-zero fractional ideals of $A$ and $I, J \in \mathcal{F}(A)$. Then:

  \[ IJ := \{ \sum_{\text{finite}} a_i b_i \mid a_i \in I, b_i \in J \} \in \mathcal{F}(A), \]

  \[ I : J := \{ k \in K \mid (k) J \subseteq I \} \in \mathcal{F}(A). \]

- **Example:** Let $p \in K$ and $J = (1, p)$. Then

  \[ A : J = \{ k \in K \mid k, kp \in A \} = \{ d \in A \mid dp \in A \} \]

  is called the ideal of the denominators of $p$.

- **Definition:** $J \in \mathcal{F}(A)$ is invertible if $\exists I \in \mathcal{F}(A)$ s.t. $IJ = A$.

- **Example:** If $J = (k), k \in K \setminus \{0\}$, then $J^{-1} = (k^{-1})$.

- **Proposition:** If $J$ is an invertible fractional ideal of $A$, then:

  \[ I = J^{-1} = A : J = \{ k \in K \mid (k) J \subseteq A \}. \]
A fractional ideal approach

- **Definition:** A fractional ideal $J$ of $A$ is **integral** if $J \subseteq A$.

- **Example:** If $p \in A$, then $J = (1, \ p) = A$. Conversely:

  $$J = (1, \ p) = (1) \Rightarrow \exists \ n \in A : \ p = n \ 1 = n \in A.$$ 

  The transfer function $p$ is **stable** iff $J = (1, \ p) = A$.

- **Definition:** A fractional ideal $J$ of $A$ is **principal** if $\exists k \in K \setminus \{0\}$:

  $$J = (k) := A \ k = \{a \ k \mid a \in A\}.$$ 

- **Example:** $J = (1, \ p)$ is principal iff there exists $k \in K \setminus \{0\}$ such that $J = (k)$, i.e., iff there exist $0 \neq d, \ n, \ x, \ y \in A$ s.t.: 

  $$\begin{align*}
  1 &= d \ k, \\
  p &= n \ k, \\
  k &= x - y \ p 
  \end{align*} \iff \begin{align*}
  k &= 1/d, \\
  p &= n/d, \\
  1/d &= x - y \ (n/d) 
  \end{align*} \iff \begin{align*}
  p &= n/d, \\
  d \ x - n \ y &= 1.
  \end{align*}$$ 

  The transfer function $p$ admits a **coprime factorization** $p = n/d$, $d \ x - n \ y = 1$, iff $J = (1/d)$, i.e., $J$ is a **principal** fractional ideal.
Example

- Let $A = H^\infty(\mathbb{C}_+)$ and $p = \frac{e^{-s}}{s-1} \in K = Q(A)$.
- Let $J = (1, p)$ be the fractional ideal of $A$ defined by 1 and $p$.
- We have $J = \left(\frac{s+1}{s-1}\right)$ since:
  \[
  \begin{align*}
  1 &= \left(\frac{s-1}{s+1}\right) \left(\frac{s+1}{s-1}\right), \\
  \frac{e^{-s}}{s-1} &= \left(\frac{e^{-s}}{s+1}\right) \left(\frac{s+1}{s-1}\right), \\
  \frac{s+1}{s-1} &= \left(1 + 2 \left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) + 2e \frac{e^{-s}}{s-1} \quad (\star).
  \end{align*}
  \]

$p = \frac{n}{d}$, $n = \frac{e^{-s}}{s+1}$, $d = \frac{s-1}{s+1}$, is a coprime factorization of $p$:

$(\star) \Leftrightarrow \left(\frac{s-1}{s+1}\right) \left(1 + 2 \left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) - \left(\frac{e^{-s}}{s+1}\right) (-2e) = 1.$
A fractional ideal approach

• Let \( p \in K \). If \( J = (1, p) \) is invertible, then \( J^{-1} = A : J \) and:

\[
1 \in J(A : J) = (1, p) (\{ d \in A | d p \in A \}) = \{ \alpha + \beta p | \alpha, \beta \in A : J \}
\]

\[
\iff \exists a, b \in A : \begin{cases} a - b p = 1, \\ a p \in A. \end{cases}
\]

If \( a \neq 0 \), then \( c = \frac{b}{a} \in K \) is s.t. \( a = \frac{1}{1-\rho c} \) and \( b = \frac{c}{1-\rho c} \) and:

\[
\left( \begin{array}{cc}
1 & -p \\
-c & 1
\end{array} \right)^{-1} = \left( \begin{array}{cc}
\frac{1}{1-\rho c} & \frac{p}{1-\rho c} \\
\frac{c}{1-\rho c} & \frac{1}{1-\rho c}
\end{array} \right) = \left( \begin{array}{cc}
a & ap \\
b & a
\end{array} \right) \in A^{2 \times 2},
\]

\( \Rightarrow c = \frac{b}{a} \) stabilizes \( p \) (\( a = 0 \Rightarrow c = 1 - b \) stabilizes \( p \)).

• If \( p \) is stabilizable, then there exists \( c \in K \) s.t.:

\[
a := \frac{1}{1-\rho c} \in A, \quad a p = \frac{p}{1-\rho c} \in A, \quad b := \frac{c}{1-\rho c} \in A.
\]

Let \( I := (a, b) \). Then, \( a - b p = 1 \in I J \Rightarrow I J = A \Rightarrow I = J^{-1}. \)
A fractional ideal approach

- **Theorem:** The following assertions are equivalent:
  1. \( p \in K := Q(A) \) is internally stabilizable.
  2. The fractional ideal \( J = (1, p) \) is invertible.
  3. \( \exists a, b \in A \) such that:
     \[
     \begin{align*}
     a - b p &= 1, \\
     a p &\in A.
     \end{align*}
     \]
  4. \( \exists b \in A \) s.t. \( b p \in A \) and \( (1 + b p) p \in A \).

- **Corollary:** If \( J = (1, p) \) is invertible, then \( J^{-1} = A : J = (a, b) \).

- \( c \in K \) internally stabilizables \( p \in K \) iff
  \[
  (1, p) \left( \frac{1}{1 - p c}, \frac{c}{1 - p c} \right) = A \iff (1, p)(1, c) = (1 - p c).
  \]
• \( c \in K := Q(A) \) stabilizes \( p \), i.e., \( c \in \text{stab}(p) \). For \( \delta \in A \), we get:

\[
(1, p)(1, c) = (1 - pc) \quad (\star)
\]

\((\star)\) \iff \[
(1, p + \delta)(1 + \delta c, c) = (1 - pc)
\]

\[
\iff (1, p + \delta) \left(1, \frac{c}{1 + \delta c}\right) = \left(1 - pc\right)
\]

\[
\iff (1, p + \delta) \left(1, \frac{c}{1 + \delta c}\right) = \left(1 - (p + \delta) \frac{c}{1 + \delta c}\right).
\]

\[
0 \quad \xrightarrow{J^{-1}} \quad J \quad \xrightarrow{(1-(p+\delta))}\quad A^2 \quad \xrightarrow{(a', b')} \quad J \quad \xrightarrow{(1-(p+\delta))}\quad 0.
\]

\[
b' = \frac{c}{1 - pc} = b, \quad a' := \frac{1 + \delta c}{1 - pc} = (1 + \delta c) a,
\]

\[
\Rightarrow \quad c' = \frac{c}{1 + \delta c} \in \text{stab}(p + \delta).
\]
The Homological Perturbation Lemma

• **Definition:** A homotopy equivalence (HE) data, denoted by 
\[(L, d) \xleftarrow{i} (M, d), h \xrightarrow{p} \]
, is defined by:

1. Two quasi-isomorphic complexes \((L, d)\) and \((M, d)\) defined by \(i\) and \(p\) (i.e., \(i\) and \(p\) induce \(H_k(L) \cong H_k(M)\)).
2. A homotopy \(h\) between \(i \circ p\) and \(\text{id}_M\) (\(i \circ p = \text{id}_M + d \circ h + h \circ d\)).

• **Definition:** A perturbation \(\delta\) of \((M, d)\) is a map on \(M\) such that:
\[(d + \delta)^2 = 0 \iff d \delta + \delta d + \delta^2 = 0\]

• **Theorem (HPL):** If the perturbation \(\delta\) is small, i.e., \(1 + \delta \circ h\) is invertible, then \((L, d') \xleftarrow{i'} (M, d + \delta), h'\) is a HE data where:

\[i' := i + h \circ (1 + \delta \circ h)^{-1} \delta \circ i, \quad p' := p + p \circ (1 + \delta \circ h)^{-1} \delta \circ h,\]
\[h' := h + h \circ (1 + \delta \circ h)^{-1} \delta \circ h, \quad d' := d + p \circ (1 + \delta \circ h)^{-1} \delta \circ i.\]

• Let us consider the case where $L = 0$.

• **Corollary:** If $(M, d)$ is a contractible complex with contraction $h$, i.e., $1 \sim 0$, i.e.,

$$h d + d h + 1 = 0,$$

and if $\delta$ is a perturbation of $b$, i.e.,

$$(d + \delta)^2 = 0,$$

such that $(1 + \delta h)$ is invertible, then $(M, d + \delta)$ is contractible with the contraction:

$$H := h (1 + \delta h)^{-1} = (1 + h \delta)^{-1} h.$$
Application 1 of the HPL to robust control

- Split short exact sequence (contractible complex): $(M, d), h.$

$$0 \rightarrow J^{-1} \xleftarrow{p \begin{pmatrix} b \\ a \end{pmatrix}} A^2 \xrightarrow{(1 - p) \begin{pmatrix} a \\ b \end{pmatrix}} J \rightarrow 0.$$  

- Perturbation $\delta$ of $(M, d)$:

$$0 \rightarrow J^{-1} \xrightarrow{\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix}} A^2 \xrightarrow{(\delta_1 - \delta_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}} J \rightarrow 0.$$  

$(\delta_1 - \delta_2) \in \text{hom}_A(A^2, J) \cong J^2,$ i.e., $\delta_1, \delta_2 \in J,$

$\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} \in \text{hom}_A(J^{-1}, A^2) \cong (\text{hom}_A(J^{-1}, A))^2 \cong J^2,$ i.e., $\delta_3, \delta_4 \in J.$
Application 1 of the HPL to robust control

- The complex $d + \delta$:

\[
0 \xrightarrow{J^{-1}} A^2 \xrightarrow{(1+\delta_1)-(p+\delta_2))} J \xrightarrow{J^{-1}} 0.
\]

\[(1 + \delta_1)(p + \delta_3) = (p + \delta_2)(1 + \delta_4), \text{ i.e., } p' := \frac{p + \delta_2}{1 + \delta_1} = \frac{p + \delta_3}{1 + \delta_4}.
\]

- Invertibility condition $(1 + \delta h)^{-1}$:

\[
1 + (\delta_1 - \delta_2) \begin{pmatrix} a \\ b \end{pmatrix} = 1 + \delta_1 a - \delta_2 b = (1 + \delta_1 - p - \delta_2) \begin{pmatrix} a \\ b \end{pmatrix} \in U(A),
\]

\[
U := l_2 + \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} (-b \quad a) = \begin{pmatrix} 1 - \delta_3 b & \delta_3 a \\ -\delta_4 b & 1 + \delta_4 a \end{pmatrix} \in \text{GL}_2(A),
\]

i.e., \( \det(U) = (1 - \delta_3 b)(1 + \delta_4 a) + \delta_3 a \delta_4 b = 1 - \delta_3 b + \delta_4 a \in U(A) \).
Application 1 of the HPL to robust control

- New contraction $H = h(1 + \delta h)^{-1}$:

\[
\begin{pmatrix}
a' \\
b'
\end{pmatrix} := \begin{pmatrix}
a \\
b
\end{pmatrix} (1 + \delta_1 a - \delta_2 b)^{-1},
\]

\[
\begin{pmatrix}
-b'' \\
a''
\end{pmatrix} := (-b \\ a) \begin{pmatrix}
1 + \delta_4 a & -\delta_3 a \\
\delta_4 b & 1 - \delta_3 b
\end{pmatrix} (1 - \delta_3 b + \delta_4 a)^{-1}
= (-b \\ a)(1 - \delta_3 b + \delta_4 a)^{-1}.
\]

- **Corollary:** For $\delta_1, \delta_2, \delta_3, \delta_4 \in J$ such that $\frac{p + \delta_2}{1 + \delta_1} = \frac{p + \delta_3}{1 + \delta_4}$ and

\[
1 + \delta_1 a - \delta_2 b \in U(A), \quad 1 - \delta_3 b + \delta_4 a \in U(A),
\]

we have the following split exact sequence:

\[
\begin{array}{c}
0 \rightarrow J^{-1} \xrightarrow{(-b'' \ a'')} A^2 \xrightarrow{(1+\delta_1 \ -(p+\delta_2))} J \xrightarrow{a' \ b'} 0.
\end{array}
\]
Application 1 of the HPL to robust control

\[(1 + \delta_1, p + \delta_2)(a', b') = A \iff \left(1, \frac{p + \delta_2}{1 + \delta_1}\right)((1 + \delta_1)a', (1 + \delta_1)b') = A.\]

**Corollary:** Let \(c = \frac{b}{a}\) be a stabilizing controller of \(p\) (i.e., \(a, b \in A, \quad a p \in A, \quad a - b p = 1\)).

For all \(\delta_1, \delta_2 \in J = (1, p)\) satisfying

\[1 + \delta_1 a - \delta_2 b \in U(A),\]

then the controller

\[
\frac{(1 + \delta_1 a - \delta_2 b)^{-1} b}{(1 + \delta_1 a - \delta_2 b)^{-1} a} = \frac{b}{a} = c
\]

stabilizes all the plants of the form:

\[p' = \frac{p + \delta_2}{1 + \delta_1}.\]
Proof 2: Direct check

\[ c = \frac{b}{a}, \quad a - b p = 1, \quad p' = \frac{p + \delta_2}{1 + \delta_1}, \quad 1 + \delta_1 a - \delta_2 b \in U(A). \]

- \( \delta_1, \delta_2 \in J \Rightarrow 1 + \delta_1, p + \delta_2 \in J \)

\[ \Rightarrow (1 + \delta_1) a, (1 + \delta_1) b, a (p + \delta_2) \in A \in A. \]

- We have:

\[ 1 - p' \ c = 1 - \frac{p + \delta_2}{1 + \delta_1} c = \frac{1 + \delta_1 - (p + \delta_2) c}{1 + \delta_1} = \frac{(a (1 + \delta_1) - (p + \delta_2) b}{a (1 + \delta_1)} \]

\[ \Rightarrow \begin{cases} 
\frac{1}{1 - p' \ c} = \frac{a (1 + \delta_1)}{1 + a \delta_1 - \delta_2 b} \in A, \\
\frac{c}{1 - p' \ c} = \frac{b (1 + \delta_1)}{1 + a \delta_1 - \delta_2 b} \in A, \\
\frac{p'}{1 - p' \ c} = \frac{(p + \delta_2) a}{1 + a \delta_1 - \delta_2 b} \in A. 
\end{cases} \]
Proof 3: Condition check

- By hypothesis, we have \( a, b \in J^{-1} \) and:

\[
(1 + \delta_1 - p - \delta_2) \begin{pmatrix} a \\ b \end{pmatrix} = (1 + \delta_1) a - (p + \delta_2) b \quad (\star)
\]

\[
= a - p b + \delta_1 a - \delta_2 b
\]

\[
= 1 + \delta_1 a - \delta_2 b \in U(A).
\]

\[
(\star) \Rightarrow \left( \frac{(1 + \delta_1) a}{1 + \delta_1 a - \delta_2 b} \right) - \left( \frac{(1 + \delta_1) b}{1 + \delta_1 a - \delta_2 b} \right) \frac{p + \delta_2}{(1 + \delta_1)} = 1.
\]

- \( \delta_1, \delta_2 \in J \Rightarrow 1 + \delta_1, p + \delta_2 \in J \)

\[
\Rightarrow (1 + \delta_1) a, (1 + \delta_1) b, a(p + \delta_2) \in A \in A.
\]

- Finally, we have:

\[
\left( \frac{(1 + \delta_1) a}{1 + \delta_1 a - \delta_2 b} \right) \frac{p + \delta_2}{(1 + \delta_1)} = \frac{a(p + \delta_2)}{1 + \delta_1 a - \delta_2 b} \in A.
\]
Proof 4: A fractional ideal check

• By hypothesis, we have:

\[(1 + \delta_1 - p - \delta_2) \begin{pmatrix} a \\ b \end{pmatrix} = (1 + \delta_1) a - (p + \delta_2) b \quad (*)\]

\[= a - p b + \delta_1 a - \delta_2 b\]

\[= 1 + \delta_1 a - \delta_2 b \in U(A).\]

• \(\delta_1, \delta_2 \in J \Rightarrow 1 + \delta_1, p + \delta_2 \in J \Rightarrow (1 + \delta_1, p + \delta_2) \subseteq J.\)

• \((a, b)(1 + \delta_1, p + \delta_2) \subseteq (a, b) J \subseteq A.\)

• \(1 + \delta_1 a - \delta_2 b = (1 + \delta_1) a - (p + \delta_2) b \in (a, b)(1 + \delta_1, p + \delta_2)\)

\[\Rightarrow (a, b)(1 + \delta_1, p + \delta_2) = A.\]
Application 1 of the HPL to robust control

- **Thm:** If $A$ be a Banach algebra, then $\| a \|_A < 1 \Rightarrow 1 - a \in U(A)$.

- $\delta_1 = \alpha_1 + \beta_1 p \in J$, $\delta_2 = -\alpha_2 - \beta_2 p \in J$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in A$.

- **Corollary:** Let $A$ be a Banach algebra, $p \in K := Q(A)$ be a stabilizable plant and $c = b/a$ a stabilizing controller of $p$, where:

  $$a, b \in A, \quad a p \in A, \quad a - b p = 1.$$ 

  Moreover, let:

  $$a = \frac{1}{1 - p c}, \quad a p = \frac{p}{1 - p c}, \quad b = \frac{c}{1 - p c}, \quad b p = \frac{p c}{1 - p c}.$$ 

  Then, for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in A$ such that

  $$\| \delta_1 a - \delta_2 b \|_A = \| \alpha_1 a + \beta_1 (a p) + \alpha_2 b + \beta_2 (b p) \|_A < 1,$$

  then $c = \frac{b}{a}$ stabilizes all the plants of the form:

  $$p' = \frac{\alpha_2 + (1 + \beta_2) p}{(1 + \alpha_1) + \beta_1 p}.$$
Application 1 of the HPL to robust control

\[ p' = \frac{\alpha_2 + (1 + \beta_2) p}{(1 + \alpha_1) + \beta_1 p}, \quad \| \alpha_1 a + \beta_1 (a p) + \alpha_2 b + \beta_2 (b p) \|_A < 1 \]

- **Multiplicative perturbation:** \( \alpha_1 = \alpha_2 = \beta_1 = 0 \),
  \[ \Rightarrow \quad c \in \text{stab}\left( \left\{ p' = (1 + \beta_2) p \mid \| \beta_2 \|_A < \| b p \|_A^{-1} \right\} \right). \]
- **Additive perturbation:** \( \alpha_1 = \beta_1 = \beta_2 = 0 \),
  \[ \Rightarrow \quad c \in \text{stab}\left( \left\{ p' = p + \alpha_2 \mid \| \alpha_2 \|_A < \| b \|_A^{-1} \right\} \right). \]
- **Inverse additive perturbation:** \( \alpha_1 = \alpha_2 = \beta_2 = 0 \),
  \[ \Rightarrow \quad c \in \text{stab}\left( \left\{ p' = \frac{p}{1 + \beta_1 p} \mid \| \beta_1 \|_A < \| a p \|_A^{-1} \right\} \right). \]
- **Inverse multiplicative perturbation:** \( \alpha_2 = \beta_1 = \beta_2 = 0 \),
  \[ \Rightarrow \quad c \in \text{stab}\left( \left\{ p' = \frac{p}{1 + \alpha_1} \mid \| \alpha_1 \|_A < \| a \|_A^{-1} \right\} \right). \]
Application 1 of the HPL to robust control

- **Theorem** (small gain theorem): If $A = H^\infty(\mathbb{C}_+)$, then $\|a\|_A < 1$ is a necessary and sufficient condition for $1 - a \in U(A)$.

- **Definition:** Let $M \in (H^\infty(\mathbb{C}_+))^{p \times q}$. Then, we define:

  $$\|M\|_\infty := \sup_{s \in \mathbb{C}_+} \sigma(M(s)).$$

- **Lemma:** $\|M_1 M_2\|_\infty \leq \|M_1\|_\infty \|M_2\|_\infty$.

  $$\|\alpha_1 a + \beta_1 (a p) + \alpha_2 b + \beta_2 (b p)\|_\infty < 1$$

  $\Leftrightarrow$ $$\|(\alpha_1 \beta_1 \alpha_2 \beta_2)\|_\infty < (\| (a \ ap \ b \ b p)^T \|_\infty)^{-1}$$

- **Parametrization of all the stabilizing controllers of $p$:**

  $$c(q) := \frac{b + q}{a + p q}, \quad \forall q \in J^{-2} = (a^2, b^2), \quad a + p q \neq 0.$$ 

  $\Rightarrow$ optimization pbs: Nehari problem, Hankel norm, Matlab, . . .
Application 2 of the HPL to robust control

- Split short exact sequence (contractible complex): \((M, d), h\).

\[
\begin{array}{cccccc}
0 & \rightarrow & I^{-1} & \xrightarrow{\begin{pmatrix} n \\ d \end{pmatrix}} & A^2 & \xrightarrow{\begin{pmatrix} d & -n \end{pmatrix}} & I & \rightarrow & 0, \\
\end{array}
\]

where \(I := (d, n)\), \(I^{-1} = \{ k \in K \mid k \cdot d, k \cdot n \in A \}\), \(d \cdot x - n \cdot y = 1\).

- Perturbation \(\delta\) of \((M, d)\):

\[
\begin{array}{cccccc}
0 & \rightarrow & I^{-1} & \xrightarrow{\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix}} & A^2 & \xrightarrow{\begin{pmatrix} \delta_1 & -\delta_2 \end{pmatrix}} & I & \rightarrow & 0. \\
\end{array}
\]

\((\delta_1 - \delta_2) \cdot \in \text{hom}_A(A^2, I) \cong I^2\), i.e., \(\delta_1, \delta_2 \in I\),

\[
\begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} \cdot \in \text{hom}_A(I^{-1}, A^2) \cong (\text{hom}_A(I^{-1}, A))^2 \cong I^2, \text{ i.e., } \delta_3, \delta_4 \in I. 
\]
Application 2 of the HPL to robust control

- The complex \( d + \delta \):

\[
0 \rightarrow I^{-1} \rightarrow A^2 \rightarrow I \rightarrow 0.
\]

\[
(d + \delta_1)(n + \delta_3) = (n + \delta_2)(d + \delta_4), \text{ i.e., } p' := \frac{n + \delta_2}{d + \delta_1} = \frac{n + \delta_3}{d + \delta_4}.
\]

- Invertibility condition \((1 + \delta h)^{-1} \):

\[
1 + (\delta_1 - \delta_2) \begin{pmatrix} x \\ y \end{pmatrix} = 1 + \delta_1 x - \delta_2 y = (d + \delta_1 - n - \delta_2) \begin{pmatrix} x \\ y \end{pmatrix} \in U(A),
\]

\[
U := l_2 + \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} (-y, x) = \begin{pmatrix} 1 - \delta_3 y & \delta_3 x \\ -\delta_4 y & 1 + \delta_4 x \end{pmatrix} \in GL_2(A),
\]

i.e., \( \det(U) = (1 - \delta_3 y)(1 + \delta_4 x) + \delta_3 x \delta_4 y = 1 - \delta_3 y + \delta_4 x \in U(A) \).
Application 2 of the HPL to robust control

- New contraction $H = h(1 + \delta h)^{-1} = (1 + h \delta)^{-1} h$:
  \[
  \begin{pmatrix}
  x' \\
  y'
  \end{pmatrix}
  :=
  \begin{pmatrix}
  x \\
  y
  \end{pmatrix}
  (1 + \delta_1 x - \delta_2 y)^{-1},
  \]
  \[
  \begin{pmatrix}
  -y'' & x''
  \end{pmatrix}
  :=
  (-y & x)
  \begin{pmatrix}
  1 + \delta_4 x & -\delta_3 x \\
  \delta_4 y & 1 - \delta_3 y
  \end{pmatrix}
  (1 - \delta_3 y + \delta_4 x)^{-1}
  =
  (-y & x)(1 - \delta_3 y + \delta_4 x)^{-1}.
  
- Corollary: For $\delta_1, \delta_2, \delta_3, \delta_4 \in J$ such that $p'$ holds and
  \[1 + \delta_1 x - \delta_2 y \in U(A), \quad 1 - \delta_3 y + \delta_4 x \in U(A),\]
  we have the following split exact sequence:
  \[
  0 \longrightarrow I^{-1} \xrightarrow{(-y' & x')} A^2 \xrightarrow{(d+\delta_1 - (n+\delta_2))} I \xrightarrow{(x'' & y'')} 0.
  \]
**Corollary:** Let $c = y/x$ be a stabilizing controller of $p = n/d$

(i.e., $x, y \in K$, $d x, n x, d y \in A$, $d x - n y = 1$).

For all $\delta_1, \delta_2 \in l = (d, n)$ satisfying

$$1 + \delta_1 x - \delta_2 y \in U(A),$$

then the controller

$$\frac{(1 + \delta_1 x - \delta_2 y)^{-1} y}{(1 + \delta_1 x - \delta_2 y)^{-1} x} = \frac{y}{x} = c$$

stabilizes all the plants of the form:

$$p' = \frac{n + \delta_2}{d + \delta_1}.$$
Application 2 of the HPL to robust control

- \( \delta_1 = \alpha_1 d + \beta_1 n \in l, \ \delta_2 = -\alpha_2 d - \beta_2 n \in l, \ \alpha_1, \alpha_2, \beta_1, \beta_2 \in A. \)

- **Corollary:** Let \( A \) be a **Banach algebra**. Then, we have:

\[
\forall \alpha_1, \alpha_2, \beta_1, \beta_2 \in A : \\
\| \delta_1 x - \delta_2 y \|_A = \| \alpha_1 (d x) + \beta_1 (n x) + \alpha_2 (d y) + \beta_2 (n y) \|_A < 1, \ (*)
\]

then \( c = \frac{y}{x} \) stabilizes all the plants of the form:

\[
p' = \frac{(1 + \beta_2) n + \alpha_2 d}{(1 + \alpha_1) d + \beta_1 n}.
\]

- If \( A = H^\infty(\mathbb{C}_+) \), then \( (*) \) is also a **necessary condition**.

\[
\| (\alpha_1 \quad \beta_1 \quad \alpha_2 \quad \beta_2) \|_\infty < \left( \| (d x \quad n x \quad d y \quad n y)^T \|_\infty \right)^{-1} \Rightarrow (\star)
\]
Application of the HPL to robust control

\[ p' = \frac{(1 + \beta_2) n + \alpha_2 d}{(1 + \alpha_1) d + \beta_1 n}, \quad \| \alpha_1 (d x) + \beta_1 (n x) + \alpha_2 (d y) + \beta_2 (n y) \|_A < 1 \]

- **Multiplicative perturbation:** \( \alpha_1 = \alpha_2 = \beta_1 = 0, \)
  \[ \Rightarrow \quad c \in \text{stab}(\{ p' = (1 + \beta_2) p \mid \| \beta_2 \|_A < \| n y \|_A^{-1} \}). \]

- **Additive perturbation:** \( \alpha_1 = \beta_1 = \beta_2 = 0, \)
  \[ \Rightarrow \quad c \in \text{stab}(\{ p' = p + \alpha_2 \mid \| \alpha_2 \|_A < \| d y \|_A^{-1} \}). \]

- **Inverse additive perturbation:** \( \alpha_1 = \alpha_2 = \beta_2 = 0, \)
  \[ \Rightarrow \quad c \in \text{stab}\left( \left\{ p' = \frac{p}{1 + \beta_1} \mid \| \beta_1 \|_A < \| n x \|_A^{-1} \right\} \right). \]

- **Inverse multiplicative perturbation:** \( \alpha_2 = \beta_1 = \beta_2 = 0, \)
  \[ \Rightarrow \quad c \in \text{stab}\left( \left\{ p' = \frac{p}{1 + \alpha_1} \mid \| \alpha_1 \|_A < \| d x \|_A^{-1} \right\} \right). \]
Corollary:

1. If \( p = n/d \) is a coprime factorization \((d \cdot x - n \cdot y = 1, \ x, \ y \in A)\), then \( \forall \ \delta_1, \delta_2 \in I = (d, \ n) = A, \ \| \delta_1 \cdot x - \delta_2 \cdot y \|_A < 1, \)

\( c = \frac{y}{x} \) stabilizes all the plants of the form \( p' = \frac{n + \delta_2}{d + \delta_1} \).

2. If \( p = n/d \) is a coprime factorization \((d \cdot x - n \cdot y = 1, \ x, \ y \in A)\),

\[ \forall q \in A, \ d \cdot (x + q \cdot n) - n \cdot (y + q \cdot d) = 1, \]

and for all \( \delta_1, \delta_2, q \in A \) such that \( \| \delta_1 \cdot (x + q \cdot n) - \delta_2 \cdot (y + q \cdot d) \|_A < 1, \) \(^\star\)

\( c(q) = \frac{y + q \cdot d}{x + q \cdot n} \) stabilizes all the plants of the form \( p' = \frac{n + \delta_2}{d + \delta_1} \).
Application 2 of the HPL to robust control

• **Theorem (Nehari):** If the coprime factorization \( p = n/d \), where \( d \times - n \times y = 1 \) is normalized, i.e., \( d^* \times d - n^* \times n = 1 \) on \( i \mathbb{R} \), then

\[
\inf_{q \in H^\infty(\mathbb{C}_+)} \| (x + q \times n \times y + q \times d)^T \|_{\infty} = \sqrt{1 + \| \Gamma_l \|_H^2},
\]

where \( l := d^* \times y - n^* \times x \in L^\infty(i \mathbb{R}) \), \( \Gamma_l \) is the Hankel operator

\[
\Gamma_l : H^2(\mathbb{C}_+) \longrightarrow H^2(\mathbb{C}_-) = H^2(\mathbb{C}_+)^T
\]

\[
\nu \longmapsto P_{H^2(\mathbb{C}_-)}(l \times u),
\]

and \( P_{H^2(\mathbb{C}_-)} : L^2(i \mathbb{R}) = H^2(\mathbb{C}_+) \oplus H^2(\mathbb{C}_-) \longrightarrow H^2(\mathbb{C}_-) \) proj..

• **Consequence:**

\[
\| (\delta_1 - \delta_2) \|_{\infty} < \frac{1}{\inf_{q \in H^\infty(\mathbb{C}_+)} \| (x + q \times n \times y + q \times d)^T \|_{\infty}} = \frac{1}{\sqrt{1 + \| \Gamma_l \|_H^2}}.
\]

• If \( \Gamma_l \) is compact (e.g., \( l \in H^\infty(\mathbb{C}_+) + C(i \mathbb{R} \cup \{\infty\}) \)), then there exists a unique \( q \in H^\infty(\mathbb{C}_+) \) for which the infimum is reached.
Classical references


References


