

Modular Metric Spaces

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ABSTRACT. We give a short introduction to the theory of modular metric spaces, including some fixed point theorems due to Chistyakov. We give new proofs of these theorems, inspired by Palais’s approach to the Banach fixed point theorem

1. INTRODUCTION

We begin with a simple motivating example. Let X be the set of all points above water on the earth’s surface. For two points x, y , let us denote by $v_t(x, y)$ the average speed required to travel directly over land from x to y in a time t . What are the properties of the function $v_t(x, y)$?

Clearly, if we fix x and y , then $v_t(x, y)$ takes nonnegative values and is a non-increasing function of t . And of course, this function is symmetric in x and y . But there is an issue we have glossed over — what if x and y lie in different landmasses? Since we are required to travel by land, it is impossible to get from x to y in a time t , no matter how fast we travel. As we would like our speed function to be defined in all circumstances, it is reasonable to allow extended real values and to assign the value $v_t(x, y) = \infty$ in this case.

To summarize, we now have a function taking $t > 0$ and $x, y \in X$ to $v_t(x, y) \in [0, \infty]$ that is symmetric in x, y and non-increasing in t . There is one further property worthy of note. A simple calculation with average speeds shows that

$$v_{s+t}(x, y) \leq v_s(x, z) + v_t(z, y)$$

for all $s, t > 0$ and all $x, y, z \in X$.

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In 2006, Vyacheslav Chistyakov [4, 2] introduced the concept of a **metric modular** on a set, inspired partly by the classical linear modulars on function spaces employed by Nakano and other in the 1950s. Our average speed function is an example of a modular in the sense of Chistyakov.

The concept of a modular on a vector space was introduced by Nakano in 1950 [10] and refined by Musielak and Orlicz in 1959 [9]. In the formulation given by Kowzslowski [7, 6], a **modular** on a vector space X is a function $m: X \rightarrow [0, \infty]$ satisfying

- (1) $m(x) = 0$ if and only if $x = 0$;
- (2) $m(ax) = m(x)$ for every $a \in \mathbb{R}$ with $|a| = 1$;
- (3) $m(ax + by) \leq m(x) + m(y)$ if $a, b \geq 0$ and $a + b = 1$.

A modular m is said to be **convex** if, instead of (3), it satisfies the stronger property

- (3') $m(ax + by) \leq am(x) + bm(y)$ if $a, b \geq 0$ and $a + b = 1$.

Given a modular m on X , the **modular space** is defined by

$$X_m = \{x \in X : m(ax) \rightarrow 0 \text{ as } a \rightarrow 0\}$$

It is possible to define a corresponding F-norm (or a norm when m is convex) on the modular space. The Orlicz spaces L^φ are examples of this construction [12].

The notion of a metric modular on an arbitrary set was introduced in 2006 by Chistyakov as a generalization of these ideas. Chistyakov's formulation provides a good framework in which to study fixed point phenomena.

2. MODULAR METRIC SPACES

We start with the definition given by Chistyakov.

Definition 1. Let X be a nonempty set. A **metric modular** on X is a function

$$w: (0, \infty) \times X \times X \rightarrow [0, \infty],$$

written as $(\lambda, x, y) \mapsto w_\lambda(x, y)$, that satisfies the following axioms:

- (1) $w_\lambda(x, y) = 0$ if and only if $x = y$ for all $\lambda > 0$ and $x, y \in X$
- (2) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$
- (3) $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If the context is clear, we refer to a metric modular simply as a **modular**. A modular w is said to be **strict** if it has the following property: given $x, y \in X$ with $x \neq y$, we have $w_\lambda(x, y) > 0$ for all $\lambda > 0$.

A modular w on X is said to be **convex** if, instead of (3), it satisfies the stronger inequality

$$w_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu}w_\lambda(x, z) + \frac{\mu}{\mu+\lambda}w_\mu(z, y), \quad (1)$$

for all $\lambda, \mu > 0$ and $x, y, z \in X$.

Let (X, d) be a metric space with at least two points. There are several ways to define a metric modular on X .

Example 2.1.

$$w_\lambda(x, y) = d(x, y)$$

In this case, property (3) in the definition of a modular is just the triangle inequality for the metric. This modular is not convex - just take $z = y$ and $\mu = \lambda$ in (1).

Example 2.2.

$$w_\lambda(x, y) = \frac{d(x, y)}{\lambda}$$

In this case, we can think of $w_\lambda(x, y)$ as the average velocity required to travel from x to y in time λ . A simple calculation with the triangle inequality shows that this modular is convex.

Example 2.3.

$$w_\lambda(x, y) = \begin{cases} \infty & \text{if } \lambda < d(x, y), \\ 0 & \text{if } \lambda \geq d(x, y) \end{cases}$$

This simple example could be seen as an extreme case of the velocity metaphor — if the time available is less than $d(x, y)$, then it is impossible to travel from x to y , but if the time is at least $d(x, y)$ then we can travel instantaneously. This modular is also convex.

We now look at some of the basic properties of modulars.

Proposition 2.4. *Let w be a modular on the set X .*

- (a) *For every $x, y \in X$, the function $\lambda \mapsto w_\lambda(x, y)$ is non-increasing.*

- (b) Let w be a convex modular. For $x, y \in X$, if $w_\lambda(x, y)$ is finite for at least one value of λ , then $w_\lambda(x, y) \rightarrow 0$ as $\lambda \rightarrow \infty$ and $w_\lambda(x, y) \rightarrow \infty$ as $\lambda \rightarrow 0+$.
- (c) If w is a convex modular, then the function $v_\lambda(x, y) = \frac{w_\lambda(x, y)}{\lambda}$ is a modular on X .

Proof.

- (a) follows from property (3) of modulars, taking $z = y$.
- (b) Taking $z = y$ in equation (1) gives

$$w_{\lambda'}(x, y) \leq \frac{\lambda}{\lambda'} w_\lambda(x, y)$$

whenever $\lambda' > \lambda$. Choosing λ for which $w_\lambda(x, y)$ is finite, we see that $w_{\lambda'}(x, y) \rightarrow 0$ as $\lambda' \rightarrow \infty$. And if we fix λ' such that $w_{\lambda'}(x, y) < \infty$, we get $w_\lambda(x, y) \rightarrow \infty$ as $\lambda \rightarrow 0+$.

(c) It is obvious that v satisfies the first two properties of a modular. Property (3) for v follows easily from the convexity condition on w_λ . \square

3. MODULAR SETS AND MODULAR CONVERGENCE

Given a modular w on X and a point x_0 in X , the two sets

$$X_w(x_0) = \{x \in X : w_\lambda(x, x_0) \rightarrow 0 \text{ as } \lambda \rightarrow \infty\}$$

and

$$X_w^*(x_0) = \{x \in X : \exists \lambda > 0 \text{ such that } w_\lambda(x, x_0) < \infty\}$$

are each known as **Modular Sets** around x_0 . These sets can be thought of as comprising all the points that are “accessible” in some sense from x_0 . In both cases, the modular sets form a partition of X . In our motivating example, the modular sets are the individual land masses.

It is clear that $X_w(x_0) \subset X_w^*(x_0)$ in general and Example 1 shows that this inclusion may be strict. While there is some ambiguity in using the same term for these two types of sets, in the sequel we shall only be working with modular sets of the form $X_w^*(x_0)$.

Proposition 3.1. *If w is convex then*

$$X_w(x_0) = X_w^*(x_0).$$

Proof. This follows immediately from Proposition 2.4 (b). \square

We now turn our attention to some notions of convergence.

Definition 2. Let w be a modular on X . A sequence (x_n) in X is said to be w -**convergent** (or **modular convergent**) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on (x_n) and x , such that $\lim_{n \rightarrow \infty} w_\lambda(x_n, x) = 0$. A sequence (x_n) in X is said to be w -**Cauchy** if there exists $\lambda > 0$, possibly depending on the sequence, such that $w_\lambda(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$. X is said to be w -**complete** if every w -Cauchy sequence is w -convergent.

Metrics on the modular set

Let w be a modular on X and let X_w be any one of the modular sets defined by w . Then the formula

$$d_w(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \geq \lambda\}, \quad \forall x, y \in X_w$$

defines a metric on X_w [4].

If the modular w is convex, then the modular space can be endowed with another a metric d_w^* given by

$$d_w^*(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq 1\}.$$

These metrics on the modular set are strongly equivalent:

$$d_w \leq d_w^* \leq 2d_w.$$

We refer to [4] for details.

The following result shows the relationship between modular and metric convergence. The proof can be found in [4].

Proposition 3.2. *Let w be a convex modular on X . Given a sequence x_n for X_w^* and an element $x \in X_w^*$, we have:*

$$\lim_{n \rightarrow \infty} d_w^*(x_n, x) = 0 \iff \lim_{n \rightarrow \infty} w_\lambda(x_n, x) = 0 \text{ for every } \lambda > 0.$$

Chistyakov gives an example to show that modular convergence is strictly weaker than metric convergence in general [3]

4. FIXED POINT THEOREMS

In 2011, Chistyakov generalized the Banach fixed point theorem to the setting of modular metric spaces. Consider the definition of a contraction on a metric space: $d(Tx, Ty) \leq kd(x, y)$ for all x, y , where k is some constant between 0 and 1. Looking at Examples 1 and 2, we see that there are at least two ways to generalise this to modular metric spaces. Chistyakov gives two definitions:

Definition 3. Let w be a modular on a set X and let X_w^* be a modular set. A mapping $T: X_w^* \rightarrow X_w^*$ is said to be **modular contractive** (or a w -contraction) if there exist numbers $k \in (0, 1)$ and $\lambda_0 > 0$ such that

$$w_{k\lambda}(Tx, Ty) \leq w_\lambda(x, y)$$

for all $0 < \lambda \leq \lambda_0$ and all $x, y \in X_w^*$.

His second definition is stronger:

Definition 4. A mapping $T: X_w^* \rightarrow X_w^*$ is said to be **strongly modular contractive** (or a *strong w -contraction*) if there exist numbers $0 < k < 1$ and $\lambda_0 > 0$ such that

$$w_{k\lambda}(Tx, Ty) \leq kw_\lambda(x, y)$$

for all $0 < \lambda \leq \lambda_0$ and all $x, y \in X_w^*$.

Chistyakov proved fixed point theorems for modular contractive and strongly modular contractive mappings. Rather than follow his proofs, our approach to these results is inspired by Richard Palais's proof of the Banach fixed point theorem [11].

Suppose that T is a contraction on a metric space (X, d) with contraction constant k . Thus, we have $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$. Combining this with an application of the triangle inequality to the points x, y, Tx and Ty , we get the inequality

$$d(x, y) \leq \frac{d(x, Tx) + d(y, Ty)}{1 - k}.$$

Palais called this the **Fundamental Contraction Inequality**. It is a key ingredient in his proof, where it is used to establish the Cauchy property for the sequence generated by iterating the mapping T on some initial point.

We begin with a variant of Palais's inequality for modular contractive mappings.

Proposition 4.1 (Fundamental Modular Contraction Inequality). *Let w be a convex modular in X , let $T: X_w^* \rightarrow X_w^*$ be a modular contractive mapping, with $w_{k\lambda}(Tx, Ty) \leq w_\lambda(x, y)$ for $0 < \lambda \leq \lambda_0$. Let $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = (1 - k)\lambda$, where $0 < \lambda < \lambda_0$. Then*

$$w_\lambda(x, y) \leq \frac{\lambda_1 w_{\lambda_1}(x, Tx) + \lambda_2 w_{\lambda_2}(y, Ty)}{\lambda(1 - k)} \quad (2)$$

for every $x, y \in X_w^*$

Proof. By the convex property, taking $\lambda = \lambda_1 + k\lambda + \lambda_2$, we get

$$w_{\lambda_1+k\lambda+\lambda_2}(x, y) \leq \frac{\lambda_1}{\lambda}w_{\lambda_1}(x, Tx) + kw_{k\lambda}(Tx, Ty) + \frac{\lambda_2}{\lambda}w_{\lambda_2}(y, Ty)$$

since $\lambda_1 + \lambda_2 = (1 - k)\lambda$ and $w_{k\lambda}(Tx, Ty) \leq w_\lambda(x, y)$.

Therefore,

$$w_\lambda(x, y) \leq \frac{\lambda_1}{\lambda}w_{\lambda_1}(x, Tx) + kw_\lambda(x, y) + \frac{\lambda_2}{\lambda}w_{\lambda_2}(y, Ty).$$

Hence

$$w_\lambda(x, y) = \frac{\lambda_1 w_{\lambda_1}(x, Tx) + \lambda_2 w_{\lambda_2}(y, Ty)}{\lambda(1 - k)}.$$

□

We now give the first fixed point theorem on modular metric spaces by Chistyakov, but our proof uses the Fundamental Modular Contraction Inequality given in the preceding proposition.

Theorem 4.2 ([3]). *Let w be a strict convex modular on X such that the modular space X_w^* is w -complete and let $T: X_w^* \rightarrow X_w^*$ be a w -contractive map such that for each $\lambda > 0$ there exists an $x = x(\lambda) \in X_w^*$ such that $w_\lambda(x, Tx) < \infty$. Then T has a fixed point x_* in X_w^* . If the modular w assumes only finite values on X_w^* , then the condition $w_\lambda(x, Tx) < \infty$ is redundant, the fixed point x_* of T is unique and for each $x_0 \in X_w^*$ the sequence of iterates $T^n x_0$ is modular convergent to x_* .*

Proof. If x, y are both fixed point then $w_\lambda(x, y) = 0$ so $x = y$. Hence a contraction mapping can have at most one fixed point. Its remains to show that for any x_0 in X the sequence $T^n(x_0)$ is Cauchy. Taking $x = T(x_0)$ and $y = T^n(x_0)$ and using the Main Contraction Modular Inequality, we get

$$\begin{aligned} w_\lambda(T^n(x_0), T^m(x_0)) &\leq \\ &\frac{\lambda_1 w_{\lambda_1}(T(T^n(x_0)), T^n(x_0)) + \lambda_2 w_{\lambda_2}(T(T^m(x_0)), T^m(x_0))}{1 - k} \\ &= \frac{\lambda_1 w_{\lambda_1}(T^n(T(x_0)), T^n(x_0)) + \lambda_2 w_{\lambda_2}(T^m(T(x_0)), T^m(x_0))}{1 - k} \\ &\leq \frac{\lambda_1 w_{k^{-n}\lambda_1}(T(x_0), x_0) + \lambda_2 w_{k^{-m}\lambda_2}(T(x_0), x_0)}{1 - k}. \end{aligned}$$

Since $k^{-n}\lambda_1 \rightarrow \infty$ as $n \rightarrow \infty$ then $k^{-n}\lambda_1 > \lambda$ if n big enough. Similar $k^{-m}\lambda_2 > \lambda$ if m big enough. Since the function $\lambda \mapsto w_\lambda(x, y)$ is non-increasing, and $0 < k < 1$ we have

$$w_{k^{-m}\lambda_2}(Tx_0, x_0), w_{k^{-n}\lambda_1}(Tx_0, x_0) \leq w_\lambda(Tx_0, x_0) < \infty.$$

Hence

$$w_\lambda(T^n(x_0), T^m(x_0)) \rightarrow 0.$$

Therefore $T^n(x_0)$ is Cauchy. \square

Chistyakov's second fixed point theorem drops the convexity assumption on the modular, replacing it with the requirement that the mapping be strongly contractive. We start with another variation of Palais's inequality for these mappings.

Proposition 4.3 (Fundamental Strong Modular Contraction Inequality). *Let w be a modular on X and let $T: X_w^* \rightarrow X_w^*$ be strongly modular contractive mapping, with $w_{k\lambda}(Tx, Ty) \leq kw_\lambda(x, y)$ for $0 < \lambda \leq \lambda_0$. Let $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = (1 - k)\lambda$, $0 < \lambda < \lambda_0$. Then*

$$w_\lambda(x, y) \leq \frac{w_{\lambda_1}(x, Tx) + w_{\lambda_2}(y, Ty)}{1 - k} \quad (3)$$

for all $x, y \in X_w^*$.

Proof. By property (3) in the definition of a modular, we get

$$w_{\lambda_1+k\lambda+\lambda_2}(x, y) \leq w_{\lambda_1}(x, Tx) + w_{k\lambda}(Tx, Ty) + w_{\lambda_2}(y, Ty)$$

Using $\lambda_1 + \lambda_2 = (1 - k)\lambda$ with the strong contractive property of T ,

$$w_\lambda(x, y) \leq w_{\lambda_1}(x, Tx) + kw_\lambda(x, y) + w_{\lambda_2}(y, Ty)$$

and so

$$w_\lambda(x, y) \leq \frac{w_{\lambda_1}(x, Tx) + w_{\lambda_2}(y, Ty)}{1 - k}$$

for all $x, y \in X_w^*$. \square

We the Fundamental Strong Contraction Modular Inequality (3) to prove the second fixed point theorem.

Theorem 4.4 ([3]). *Let w be a strict modular on X such that X_w is w -complete, let $T: X_w^* \rightarrow X_w^*$ be a strongly w -contractive map such that for each $\lambda > 0$ there exists an $x = x(\lambda) \in X_w^*$ such that $w_\lambda(x, Tx) < \infty$ holds. Then T has a fixed point x_* in X_w^* . If, in addition, w is finite valued on X_w^* , then the condition $w_\lambda(x, Tx) < \infty$ is redundant, the fixed point x_* of T is unique and for each $x_0 \in X_w^*$ the sequence of iterates $T^n x_0$ is modular convergent to x_* [3].*

Proof. If x, y are both fixed points then $w_\lambda(x, y) = 0$ so $x = y$. Hence a contraction mapping can have at most one fixed point. It remains to show that for any x_0 in X the sequence $T^n(x_0)$ is Cauchy. Taking $x = T(x_0)$ and $y = T^n(x_0)$,

$$\begin{aligned} w_\lambda(T^n(x_0), T^m(x_0)) &\leq \\ &\frac{w_{\lambda_1}(T(T^n(x_0)), T^n(x_0)) + w_{\lambda_2}(T(T^m(x_0)), T^m(x_0))}{1 - k} \\ &= \frac{w_{\lambda_1}(T^n(T(x_0)), T^n(x_0)) + w_{\lambda_2}(T^m(T(x_0)), T^m(x_0))}{1 - k} \\ &\leq \frac{k^n w_{k^{-n}\lambda_1}(T(x_0), x_0) + k^m w_{k^{-m}\lambda_2}(T(x_0), x_0)}{1 - k}. \end{aligned}$$

If $k^{-n}\lambda_1 \rightarrow \infty$ then $k^{-n}\lambda_1 > \lambda$ if n is big enough. Similarly, $k^{-m}\lambda_2 > \lambda$ if n also is big enough.

Since the function $\lambda \mapsto w_\lambda(x, y)$ is non-increasing, and $0 < k < 1$ we have

$$w_{k^{-m}\lambda_2}(Tx_0, x_0), w_{k^{-n}\lambda_1}(Tx_0, x_0) \leq w_\lambda(Tx_0, x_0) < \infty.$$

Hence $w_\lambda(T^n(x_0), T^m(x_0)) \rightarrow 0$. Therefore $T^n(x_0)$ is Cauchy. \square

5. APPLICATIONS

Electrorheological fluids are liquids that rapidly solidify in the presence of an electric field. They are often studied using Sobolev spaces with a variable exponent [8]. It has been suggested that modular metric spaces may be useful in modelling them [1].

Recent work indicates that modular metric space fixed point results are well adapted to certain types of differential equations [4]. Finally, we refer to [5] for a detailed study of nonlinear superposition operators on modular metric spaces of functions.

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