Primary and Other Decompositions of Binomial Ideals

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Setting

In this talk we are concerned with ideals in the polynomial ring

$$S := \mathbb{k}[x_1, \dots, x_n],$$

with $\mathbb{k}$ a field of characteristic zero, algebraically closed "when needed".

- An ideal I is an additive subgroup of S such that for all $s \in S$ and $f \in I$ one has $fs \in I$.
- The quotient ring is S/I.
- A zerodivisor $f \in S/I$ if $\exists g \neq 0 \in S/I$ with $gf = 0$.
- The saturation $(I : J^\infty)$ of I with respect to J is

$$(I : J^\infty) := \{f \in S : \exists g \in J, g^m f \in I\}.$$

Setting II

- An ideal P is called prime if every $f \in S/P$ is a non-zerodivisor i.e. $(I : f^\infty) = I$.
- It is called primary if every $f \in S/P$ is either a non-zerodivisor or nilpotent, i.e. $f^k \in P$.
- A prime P is associated to an ideal I if $P = (I : f) := \{g \in S : fg \in I\}$.
- A primary decomposition is writing I as

$$I = P_1 \cap P_2 \cap \ldots \cap P_r.$$

with primary ideals P_i.

Binomial Ideals

- A binomial ideal is an ideal whose generators can be chosen to be differences of monomials.
- A pure difference ideal is an ideal whose generators are differences of monic monomials.

Monomial Notation : $x^m = \prod_{i=1}^n x_i^{m_i}$.

Binomial Ideals are combinatorial

A binomial is characterized by a vector $m \in \mathbb{Z}^n$ and a coefficient $c \in \Bbbk$:

$$x^{m^+} - cx^{m^-}$$

with $m_i^+ := \max\{0, m_i\}$ so that $m = m^+ - m^-$.

Theory of Binomial Ideals

The theory of binomial ideals was developed in

"Binomial Ideals" (Eisenbud / Sturmfels, 1996).

Again Binomial:

- Gröbner bases are binomial.
- Associated Primes and the radical are binomial.
- Primary components can be chosen binomial.

However, quotients of binomial ideals by monomial ideals, or single binomials need not be binomial. Intersections are not binomial.

Field requirements

Each binomial ideal has a primary decomposition into binomial ideals, if $\mathbb{k}$ is algebraically closed:

Primary decomposition over $\mathbb{Q}$

$$\langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x^2 + x + 1 \rangle$$

Primary decomposition over $\mathbb{C}$

$$\langle x^3 - 1 \rangle = \langle x - 1 \rangle \cap \langle x - \xi_3 \rangle \cap \langle x - \xi_3^2 \rangle$$

Today we only mean the second one.

Examples

- *Toric ideals* are binomial prime ideals $\rightarrow$ discrete models in Algebraic Statistics.
- *Conditional independence ideals* characterize probability distributions with certain independence relations.
- *Semigraphoid ideals* are tools for the formal manipulation of statements using inference rules.
- *Commutative semigroup rings* are quotients of S by pure difference ideals.
- *Lattice ideals* are defined by ...

Partial characters

Definition

- A partial character is a pair $(\mathcal{L}, \sigma)$, with $\mathcal{L} \subseteq \mathbb{Z}^n$ an integer lattice, $\sigma : \mathcal{L} \to \Bbbk^*$ a homomorphism.
- For each integer lattice $\mathcal{L} \subseteq \mathbb{Z}^n$, we define its saturation

$$\mathrm{Sat}(\mathcal{L}) := \{m \in \mathbb{Z}^n : dm \in \mathcal{L} \text{ for some } d \in \mathbb{Z}\}.$$

A partial character $(\mathcal{L}, \sigma)$ is called saturated if $\mathcal{L} = \mathrm{Sat}(\mathcal{L})$, and it is called a saturation of a partial character $(\mathcal{L}', \sigma')$, provided that $\mathcal{L} = \mathrm{Sat}(\mathcal{L}')$ and $\sigma'(l) = \sigma(l)$, $\forall l \in \mathcal{L}'$.

Lattice Ideals

This data defines a lattice ideal:

Definition (Lattice ideal)

$$I_+(\sigma) := \left\langle x^{m^+} - \sigma(m)x^{m^-} : m \in \mathcal{L} \right\rangle .$$

Minimal generators of lattice ideals can be computed via saturation. If B is a lattice basis then

$$I_+(\sigma) = \left\langle x^{b^+} - \sigma(b)x^{b^-} : b \in B \right\rangle : \left(\prod_{i=1}^{n} x_i \right)^{\infty} .$$

Trivial character & saturated lattice: pure toric ideals $\Longrightarrow$ 4ti2.

Cellular Ideals

The next class of binomial ideals “lattice ideals + some monomials”

Definition

- An ideal I is called *cellular* if every variable is either a non-zerodivisor or nilpotent in S/I.

Call $\mathcal{E}$, the set of cellular variables, that is the non-zerodivisors. Since every other variable is nilpotent, for an exponent vector $d = (d_i)_{i \notin \mathcal{E}}$.

$$M(\mathcal{E})^d := \left\langle x_i^{d_i} : i \notin \mathcal{E} \right\rangle$$

is contained in I.

Cellular Ideals II

Lemma

An ideal I is cellular iff, exist $\mathcal{E} \subseteq \{1, \dots, n\}$, and $d = (d_i)_{i \notin \mathcal{E}}$:

$$I = \left(I + M(\mathcal{E})^d\right) : \left(\prod_{i \in \mathcal{E}} x_i\right)^{\infty}.$$

The variety of a cellular ideal has its points in the algebraic torus

$$(\Bbbk^*)^{\mathcal{E}} := \{(x_1, \dots, x_n) \in \Bbbk^n : x_i \neq 0, i \in \mathcal{E} \text{ and } x_j = 0, \forall j \notin \mathcal{E}\}.$$

Every further analysis of a binomial variety starts with a cellular decomposition.

Approximating ideals

This can be achieved by Noetherian approximation:

Lemma (ES96, Proposition 7.2)

Let I be an ideal in a Noetherian ring S and $g \in S$ such that $(I : g) = (I : g^\infty)$. Then

1. $I = (I : g) \cap (I + \langle g \rangle)$.
2. $\mathrm{Ass}(S/(I : g)) \cap \mathrm{Ass}(S/(I + \langle g \rangle)) = \emptyset$.
3. *A minimal primary decomposition of I consists of the primary components of $(I : g)$ and those primary components of $I + \langle g \rangle$ that correspond to associated primes of I.*

Cellular decomposition

This can be turned into a very simple algorithm:
Input: I, a binomial ideal.
Output: A cellular decomposition of I

1. If I is cellular, return I.
2. Else, choose a variable which is a zerodivisor but not nilpotent modulo I.
3. Determine the power s such that $(I : x_i^s) = (I : x_i^\infty)$.
4. Iterate with $(I : x_i^s)$ and $I + \langle x_i^s \rangle$.

Termination of this algorithm is ensured since S is Noetherian and the two new ideals are properly containing I.

Minimal Primes

A radical cellular ideal is of the form $M(\mathcal{E}) + I_+(\sigma)$, its associated primes are given by saturations of σ.

Idea: Modify cellular decomposition to directly decompose the radical of I.

1. If I is cellular, collect its minimal primes.
2. Else, choose a variable which is a zerodivisor but not nilpotent modulo I.
3. Iterate with $(I : x_i^\infty)$ and $I + \langle x_i \rangle$.

Commuting Birth and Death Ideals

Everything started with a computational challenge from

“Commuting birth-death processes” (Evans/Sturmfels/Uhler ’08)

Current general-purpose implementations of primary decomposition are not able to perform the decomposition for the 2x2 grid.

The new Macaulay 2 package *Binomials* is able to do it...

... and more

A non-radical Commuting Birth and Death Ideal

$$\begin{aligned}
I^{(2,3)} := \langle U_{00}R_{01} - R_{00}U_{10}, \quad & R_{01}D_{11} - D_{01}R_{00}, \quad D_{11}L_{10} - L_{11}D_{01}, \\
L_{10}U_{00} - U_{10}L_{11}, \quad & U_{01}R_{02} - R_{01}U_{11}, \quad R_{02}D_{12} - D_{02}R_{01}, \\
D_{12}L_{11} - L_{12}D_{02}, \quad & L_{11}U_{01} - U_{11}L_{12}, \quad U_{02}R_{03} - R_{02}U_{12}, \\
R_{03}D_{13} - D_{03}R_{02}, \quad & D_{13}L_{12} - L_{13}D_{03}, \quad L_{12}U_{02} - U_{12}L_{13}, \\
U_{10}R_{11} - R_{10}U_{20}, \quad & R_{11}D_{21} - D_{11}R_{10}, \quad D_{21}L_{20} - L_{21}D_{11}, \\
L_{20}U_{10} - U_{20}L_{21}, \quad & U_{11}R_{12} - R_{11}U_{21}, \quad R_{12}D_{22} - D_{12}R_{11}, \\
D_{22}L_{21} - L_{22}D_{12}, \quad & L_{21}U_{11} - U_{21}L_{22}, \quad U_{12}R_{13} - R_{12}U_{22}, \\
R_{13}D_{23} - D_{13}R_{12}, \quad & D_{23}L_{22} - L_{23}D_{13}, \quad L_{22}U_{12} - U_{22}L_{23}\rangle
\end{aligned}$$

is not radical, but the intersection of 2638 primary components, 10 of which are not prime.

Conclusion

Implementation problems

- Compute $I : (\prod_{i\in\mathcal{E}} x_i)$ faster! $\rightarrow$ 4ti2
- Everything for finite fields.

Theoretical problem

What is the “finest” decomposition of a pure difference ideal into binomial ideals defined over $\mathbb{Q}$?

Please download and use *Binomials*!

`http://personal-homepages.mis.mpg.de/kahle/bpd/`