Abelian ideals in a Borel subalgebra of a complex simple Lie algebra

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Basic definitions

- Let $\mathfrak{g}$ denote a complex simple Lie algebra and $\mathfrak{b}$ a fixed Borel subalgebra (i.e. a maximal solvable subalgebra of $\mathfrak{g}$)
- $\mathfrak{g}$ can be decomposed as follows

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is the Cartan subalgebra and Φ the set of roots of $\mathfrak{g}$

There exists a set of roots, $\Pi := \{\alpha_1, \cdots, \alpha_r\}$ which form a basis for Φ^+ such that any other root α can be expressed as $\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$ where all the n_i^α are either all non-negative, in which case we call α a positive root, or all non-positive in which case we call α a negative root. The elements of Π are known as simple roots.

- The height of a root α is defined as, $ht(\alpha) = \sum_{i=1}^{r} n^{\alpha_i}$
- There exists a unique root $\tilde{\alpha}$ such that $ht(\tilde{\alpha}) > ht(\alpha)$ for all other $\alpha \in \Phi$, called the highest root.

- The fundamental weights $\{\omega_1, \cdots, \omega_r\}$ are defined by the condition;

$$(\omega_i, 2\alpha_j) := (\alpha_i, \alpha_j)\delta_{ij}$$

for all i and j, where $(,)$ is an invariant inner product normalized so that $(\tilde{\alpha}, \tilde{\alpha}) = 2$. This inner product is related to that induced on E from the killing form by $(,) := \frac{1}{2g} <,>$, and the number g is known as the dual Coxeter number.

Tools

The following lemma is a basic tool for investigations on root systems.

Lemma

Given two non proportional roots $\alpha, \beta \in \Phi$ then
If $(\alpha, \beta) > 0$ the $\alpha - \beta$ is a root.
If $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.
If $(\alpha, \beta) = 0$ and $\alpha + \beta \in \Phi$ then $\alpha - \beta \in \Phi$

Abelian Ideals as subsets of Φ^+

Let $\mathfrak{a} \subset \mathfrak{b}$ be an abelian ideal of a Borel subalgebra $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$. Since $\mathfrak{a}$ is an ideal it is ad-$\mathfrak{h}$ stable and hence compatible with the root space decomposition. Since $\mathfrak{a}$ is abelian, it lies inside the nilpotent radical $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, hence $\mathfrak{a}$ is of the form $\mathfrak{a} = \bigoplus_{\psi \in \Psi} \mathfrak{g}_\psi$ for some subset $\Psi \subseteq \Phi^+$. Using the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ we see

- (i) The ideal property for $\mathfrak{a}$ translates into the condition for Ψ that, $\Psi + \Phi^+ := (\Psi + \Phi^+) \cap \Phi^+ \subseteq \Psi$
- (ii) The abelian condition becomes $\Psi + \Psi := (\Psi + \Psi) \cap \Phi^+ = \emptyset$.

We now have a bijection between abelian ideals $\mathfrak{a} \subset \mathfrak{b}$ and subsets $\Psi \subset \Phi +$ that satisfy (i) and (ii).

Illustration

To illustrate this consider the G_2 root system, show below

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^{\alpha}\alpha_i$	$\sum_{i=1}^{r} m_i^{\alpha}\omega_i$
1	1	$(1,0)$	$(2,-1)$
2	1	$(0,1)$	$(-3,2)$
3	2	$(1,1)$	$(-1,1)$
4	3	$(2,1)$	$(1,0)$
5	4	$(3,1)$	$(3,-1)$
6	5	$(3,2)$	$(0,1)$

Table: G_2

Illustration

- The set $\{\alpha_1, \alpha_2, \alpha_3\}$ is not abelian since $\alpha_1 + \alpha_2 = \alpha_3 \in \Phi^+$
- The set $\{\alpha_6, \alpha_5, \alpha_4\}$ is abelian since $n_1^\alpha \geq 2$.
- The set $I := \{\alpha_1, \alpha_6\}$ is not an ideal, since $\alpha_1 + \alpha_2 \in \Phi^+$ and $\alpha_1 + \alpha_2 \notin I$.
- The set $J := \{\alpha_5, \alpha_6\}$ is an ideal, since if $\alpha + J \in \Phi^+$, then $\alpha + J \in J$.

Example of the difficulty

The root systems of most classical and exceptional Lie algebras contain far more roots than the above example. We provide a graphical illustration of this for the exceptional Lie algebra of E_6 which has 36 positive roots.

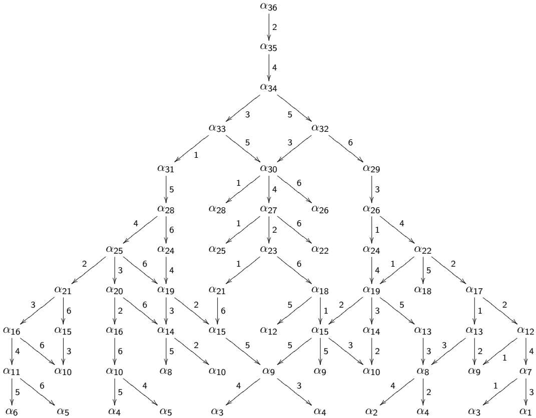

Figure: The Root Tree of E_6

Our results.

Theorem

Given a root system of a type H Lie algebra such that $n_i^{\tilde{\alpha}} = 1$, we define the set $I := \{\alpha \in \Phi^+ | n_i^\alpha = 1\}$. Then $\mathfrak{a}_I := \bigoplus_{\alpha \in I} \mathfrak{g}_\alpha$ is a maximal abelian ideal of dimension $g(\omega_i, \omega_i)$, where g is the dual Coxeter number.

Theorem

Given a root system of a type H Lie algebra other than B_n, let k be a persistent index of highest order, then the abelian ideal $\mathfrak{a}_I$ where $I = \{\alpha \in \Phi^+ | n_k^\alpha = 1\}$ is of maximum dimension.

Computational example

Let $\mathfrak{g}$ be the Lie algebra, E_6 with 36 positive roots, shown below. We now apply the two theorems above to find a maximal abelian ideals in E_6 of maximum dimension

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^{\alpha}\alpha_i$	$\sum_{i=1}^{r} m_i^{\alpha}\omega_i$
1	1	$(1,0,0,0,0,0)$	$(2,0,-1,0,0,0)$
2	1	$(0,1,0,0,0,0)$	$(0,2,0,-1,0,0)$
3	1	$(0,0,1,0,0,0)$	$(-1,0,2,-1,0,0)$
4	1	$(0,0,0,1,0,0)$	$(0,-1,-1,2,-1,0)$
5	1	$(0,0,0,0,1,0)$	$(0,0,0,-1,2,-1)$
6	1	$(0,0,0,0,0,1)$	$(0,0,0,0,-1,2)$
7	2	$(1,0,1,0,0,0)$	$(1,0,1,-1,0,0)$
8	2	$(0,1,0,1,0,0)$	$(0,1,-1,1,-1,0)$
9	2	$(0,0,1,1,0,0)$	$(-1,-1,1,1,-1,0)$
10	2	$(0,0,0,1,1,0)$	$(0,-1,-1,1,1,-1)$
11	2	$(0,0,0,0,1,1)$	$(0,0,0,-1,1,1)$
12	3	$(1,0,1,1,0,0)$	$(1,-1,0,1,-1,0)$
13	3	$(0,1,1,1,0,0)$	$(-1,1,1,0,-1,0)$
14	3	$(0,1,0,1,1,0)$	$(0,1,-1,0,1,-1)$
15	3	$(0,0,1,1,1,0)$	$(-1,-1,1,0,1,-1)$

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^\alpha \alpha_i$	$\sum_{i=1}^{r} m_i^\alpha \omega_i$
16	3	$(0,0,0,1,1,1)$	$(0,-1,-1,1,0,1)$
17	4	$(1,1,1,1,0,0)$	$(1,1,0,0,-1,0)$
18	4	$(1,0,1,1,1,0)$	$(1,-1,0,0,1,-1)$
19	4	$(0,1,1,1,1,0)$	$(-1,1,1,-1,1,-1)$
20	4	$(0,1,0,1,1,1)$	$(0,1,-1,0,0,1)$
21	4	$(0,0,1,1,1,1)$	$(-1,-1,1,0,0,1)$
22	5	$(1,1,1,1,1,0)$	$(1,1,0,-1,1,-1)$
23	5	$(1,0,1,1,1,1)$	$(1,-1,0,0,0,1)$
24	5	$(0,1,1,2,1,0)$	$(-1,0,0,1,0,-1)$
25	5	$(0,1,1,1,1,1)$	$(-1,1,1,-1,0,1)$
26	6	$(1,1,1,2,1,0)$	$(1,0,-1,1,0,-1)$
27	6	$(1,1,1,1,1,1)$	$(1,1,0,-1,0,1)$
28	6	$(0,1,1,2,1,1)$	$(-1,0,0,1,-1,1)$
29	7	$(1,1,2,2,1,0)$	$(0,0,1,0,0,-1)$

No.	ht	$\alpha = \sum_{i=1}^{r} n_i^{\alpha} \alpha_i$	$\sum_{i=1}^{r} m_i^{\alpha} \omega_i$
30	7	$(1,1,1,2,1,1)$	$(1,0,-1,1,-1,1)$
31	7	$(0,1,1,2,2,1)$	$(-1,0,0,0,1,0)$
32	8	$(1,1,2,2,1,1)$	$(0,0,1,0,-1,1)$
33	8	$(1,1,1,2,2,1)$	$(1,0,-1,0,1,0)$
34	9	$(1,1,2,2,2,1)$	$(0,0,1,-1,1,0)$
35	10	$(1,1,2,3,2,1)$	$(0,-1,0,1,0,0)$
36	11	$(1,2,2,3,2,1)$	$(0,1,0,0,0,0)$

By using 3.1 and 3.2 we find two distinct maximal abelian ideals (of equal dimension), whose dimension is maximal.

- $\Psi_1 = \{\alpha_1, \alpha_7, \alpha_{12}, \alpha_{17}, \alpha_{18}, \alpha_{22}, \alpha_{23}, \alpha_{26}, \alpha_{27}, \alpha_{29}, \alpha_{30}, \alpha_{32}, \alpha_{33}, \alpha_{34}, \alpha_{35}, \alpha_{36}\}$
- $\Psi_2 = \{\alpha_6, \alpha_{11}, \alpha_{16}, \alpha_{20}, \alpha_{21}, \alpha_{23}, \alpha_{25}, \alpha_{27}, \alpha_{28}, \alpha_{30}, \alpha_{31}, \alpha_{32}, \alpha_{33}, \alpha_{34}, \alpha_{35}, \alpha_{36}\}$

Both of which have dimension 16.

Sum Formula

Let $\mathfrak{g}$ be a simple Lie algebra, if we grade with $\tilde{\alpha}$ then we have the following two results, which can be used to construct maximal abelian ideals.

Theorem

Let $\mathfrak{a}_I$ be a maximal abelian ideal in a simple Lie algebra $\mathfrak{g}$. Then for all α such that $\alpha \in \mathfrak{g}_{(1)}$, either $\alpha \in I$ or $\tilde{\alpha} - \alpha \in I$

Theorem

Given a maximal abelian ideal I, let $\alpha \in \mathfrak{g}_{(1)}^{T}$. If $\alpha \notin I$ then

$$2n_i^{\alpha} - n_i^{\tilde{\alpha}} < 0$$

for some i.

Illustration

Using the criteria set out above we can form the following sets I (where $\mathfrak{a}_I$ is a maximal abelian ideal), by excluding one root $\alpha \in \mathfrak{g}_{(1)}$. We do this by looking for an index k such that $n_k^{\tilde{\alpha}} = 2n+1$ then form a new I for each $\alpha \in \mathfrak{g}_{(1)}^T$ such that $n_k^{\alpha} = n$ that we exclude. Below we have listed the α we will exclude from and I and the corresponding α we must include in I by the above theorem. The highest root for E_7 is $\tilde{\alpha} = (2,2,3,4,3,2,1)$. Beside each set I we provide its corresponding grading.

Illustration

- I_1: $\alpha_{53} \notin I_1$ (the highest $\alpha \in \mathfrak{g}^T_{(1)}$ such that $n^\alpha_7 = 0$) hence $\alpha_{34} \in I_1$. Grading positions $\{7\}$.
- I_2: $\alpha_{50} \notin I_1$ (the second highest $\alpha \in \mathfrak{g}^T_{(1)}$ such that $n^\alpha_7 = 0$) hence $\alpha_{39} \in I_2$. Grading positions $\{2, 7\}$.
- I_3: $\alpha_{46} \notin I_1$ (the third highest $\alpha \in \mathfrak{g}^T_{(1)}$ such that $n^\alpha_7 = 0$) hence $\alpha_{44} \in I_3$. Grading positions $\{4, 7\}$.
- I_4: $\alpha_{47} \notin I_1$ (the highest $\alpha \in \mathfrak{g}^T_{(1)}$ such that $n^\alpha_5 = 1$) hence $\alpha_{43} \in I_4$. Grading positions $\{5\}$.
- I_5: $\alpha_{52} \notin I_1$ (the highest $\alpha \in \mathfrak{g}^T_{(1)}$ such that $n^\alpha_3 = 1$) hence $\alpha_{37} \in I_5$. Grading positions $\{3\}$.
- I_6: $\alpha_{48} \notin I_1$ (the second highest $\alpha \in \mathfrak{g}^T_{(1)}$ such that $n^\alpha_3 = 1$) hence $\alpha_{42} \in I_6$. Grading positions $\{3, 6\}$.
- I_7: $\{\alpha \in \mathfrak{g}^T_{(1)}\} \in I_7$ hence $\{\alpha \in \mathfrak{g}^B_{(1)} \notin I_7\}$. Grading positions $\{3, 5, 7\}$.

The above method will also determine the number of maximal abelian ideals for a given Lie algebra $\mathfrak{g}$.

Questions

Thank You:
Any Questions/Comments

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