Computational homology in dynamical systems

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Outline

- Dynamical systems
- Rigorous numerics of dynamical systems
- Homological invariants of dynamical systems
- Computing homological invariants
- Homology algorithms for subsets of $\mathbb{R}^d$
- **Homology algorithms for maps of subsets of $\mathbb{R}^d$**
- Applications

Computing Homology of Maps

In principle, computing homology of a map $f : X \to Y$ is a three step procedure:

(1) Find a finite representation of f
(2) Use it to build the chain map
(3) Compute the map in homology from the chain map

Problems:

- What constitutes an acceptable finite representation?
- How one can build it?
- An acceptable finite representation must at least
  - be construable from an algorithm approximating values of f
  - carry the information about the homology map in the form of an easily construable chain map with the proper homology

Simplicial Approximation as Finite Representation?

- K — simplicial complex
- K_0 — the sets of veritces of K
- $\operatorname{St} v$ — the star of a vertex $v \in K$, i.e. the union of the interiors of the simplexes in K to which v belongs
- $\operatorname{sd}^n K$ — the nth barycientric subdivision of K
- $f : |K| \to |L|$ — a continuous map
- star condition

$$\forall v \in K_0 \exists w \in L_0 f(\operatorname{St} v) \subset \operatorname{St} w$$

A simplicial map $f_0 : K_0 \to L_0$ is a simplicial approximation of f if $f(\operatorname{St} v) \subset \operatorname{St} f_0(v)$ for each $v \in K_0$.

Theorem.

- Simplicial approximation exists iff f satisfies star condition.
- There exists $n \in \mathbb{N}$ such that $f : |\operatorname{sd}^n K| \to |L|$ satisfies star condition

Simplicial Approximation Algorithm?

(1) Construct a covering $\mathcal{A} := \{ A_w \mid w \in L \}$ of $|K|$ such that

$$A_w \subset f^{-1}(\operatorname{St} w)$$

(2) Find λ such that any set of diameter less than λ is contained in an element of $\mathcal{A}$ (the Lebesgue number of $\mathcal{A}$)

(3) Replace K with $\operatorname{sd}^N K$, where N is chosen so that each simplex in $\operatorname{sd}^N K$ has diameter less than $\lambda/2$.

(4) Build simplicial approximation.

Problems with Simplicial Approximation.

- In general finding approximation of f^{-1} if only a way of approximating f is given may be difficult.
- Finding A_w requires constructing approximations of $f^{-1}(\mathrm{St}\, w)$ from below, which is hard even for simple sets like balls or rectangles.
- The approximations A_w need to be good enough to form a covering of $|K|$.
- The classical proof of the existence of the Lebesgue number is non constructive.
- In general the algorithm requires substantial subdivision
- Advantage: once a simplicial approximation is found, obtaining chain map is straightforward.

There is no counterpart of simplicial approximation in cubical homology.

Cubical multivalued maps

- for an elementary interval I put

$$\overset{\circ}{I} := \begin{cases} (l, l+1) & \text{if } I = [l, l+1], \\ [l] & \text{if } I = [l, l]. \end{cases}$$

- for an elementary cube $Q = I_1 \times I_2 \times \ldots \times I_d \subset \mathrm{R}^d$ define the associated elementary cell as

$$\overset{\circ}{Q} := \overset{\circ}{I}_1 \times \overset{\circ}{I}_2 \times \ldots \times \overset{\circ}{I}_d.$$

Let X and Y be cubical sets. A multivalued map $F : X \rightrightarrows Y$ is cubical if

(1) for every $x \in X$, $F(x)$ is a cubical set,

(2) for every $Q \in \mathcal{K}(X)$, $F|_{\overset{\circ}{Q}}$ is constant, that is, if $x, x' \in \overset{\circ}{Q}$, then $F(x) = F(x')$.

Cubical multivalued maps

- $\mathcal{X}, \mathcal{Y} \subset \mathcal{K}_d$
- $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ — a multivalued combinatorial map
- Define the multivalued maps $\lfloor \mathcal{F} \rfloor, \lceil \mathcal{F} \rceil : |\mathcal{X}| \rightrightarrows |\mathcal{Y}|$ by

$$\begin{aligned} \lfloor \mathcal{F} \rfloor (x) &:= \bigcap \{ |\mathcal{F}(Q)| \mid x \in Q \in \mathcal{X} \} , \\ \lceil \mathcal{F} \rceil (x) &:= \bigcup \{ |\mathcal{F}(Q)| \mid x \in Q \in \mathcal{X} \} . \end{aligned}$$

Theorem. The maps $\lfloor \mathcal{F} \rfloor$ and $\lceil \mathcal{F} \rceil$ are cubical. The map $\lfloor \mathcal{F} \rfloor$ is lower semicontinuous and the map $\lceil \mathcal{F} \rceil$ is upper semicontinuous.

Chain selectors

- $F : X \rightrightarrows Y$ is acyclic-valued if for every $x \in X$ the set $F(x)$ is acyclic
- $\varphi : C(X) \to C(Y)$ is a chain selector of F iff

$$
\begin{aligned}
|\varphi(\widehat{Q})| &\subset F(\overset{\circ}{Q}) \text{ for all } Q \in \mathcal{K}(X), \\
\varphi(\widehat{Q}) &\in \widehat{\mathcal{K}}_0(F(Q)) \text{ for any vertex } Q \in \mathcal{K}_0(X).
\end{aligned}
$$

Theorem. (Allili, Kaczynski, 2000) Every lower semicontinuous, acyclic-valued cubical map admits a chain selector and any two such chain selectors are chain homotopic.

For a lower semicontinuous, acyclic-valued cubical map $F : X \rightrightarrows Y$ we put $H_*(F) := H_*(\varphi)$ for any chain selector φ of F.

- proof is constructive
- Mazur, Szybowski (1999) — implementation of an algorithm based on the proof
- algorithm slow — huge linear systems need to be solved

- For a bounded set $A \subset \mathbb{R}^d$ let $\mathrm{ch}\,(A)$ be the smallest cubical set containing A.
- $F : X \rightrightarrows Y$ is a representation of a continuous map $f : X \to Y$ if F is a lower semicontinuous multivalued cubical map and $f(x) \in F(x)$ for every $x \in X$.

Theorem. The map

$$M_f : X \ni x \to \mathrm{ch}\,(f(\mathrm{ch}\,(x))) \subset Y$$

is a representation of $f : X \to Y$

Rescalling

For $\alpha \in \mathbb{N}^d$ put

$$\Lambda^\alpha : \mathbb{R}^d \ni x \longrightarrow (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_d x_d) \in \mathbb{R}^d$$
$$\Omega^\alpha : \mathbb{R}^d \ni x \longrightarrow (\alpha_1^{-1} x_1, \alpha_2^{-1} x_2, \ldots, \alpha_d^{-1} x_d) \in \mathbb{R}^d$$
$$X^\alpha := \Lambda^\alpha(X)$$
$$\Lambda^\alpha_X : X \ni x \longrightarrow \Lambda^\alpha(x) \in X^\alpha$$
$$f^\alpha := f \circ \Omega^\alpha$$

Theorem.

- $M_{\Lambda^\alpha_X}$ is acyclic valued
- M_{f^α} is acyclic valued for sufficiently large α
- $H_*(M_{f^\alpha}) \circ H_*(M_{\Lambda^\alpha_X})$ does not depend on α for large α

For a continuous $f : X \longrightarrow Y$ we put

$$H_*(f) := H_*(M_{f^\alpha}) \circ H_*(M_{\Lambda^\alpha_X})$$

for a sufficiently large α.

Graph approach

Theorem. If $F : X \rightrightarrows Y$ is an upper semicontinuous cubical map, then

$$G(F) := \{ (x, y) \in X \times Y \mid y \in F(x) \}$$

is a cubical set.

- $p : X \times Y \to X$, $q : X \times Y \to Y$ — projections
- p and q induce chain maps
- by Vietoris Theorem $H_*(p)$ is an isomorphism if F is acyclic valued

Graph approach

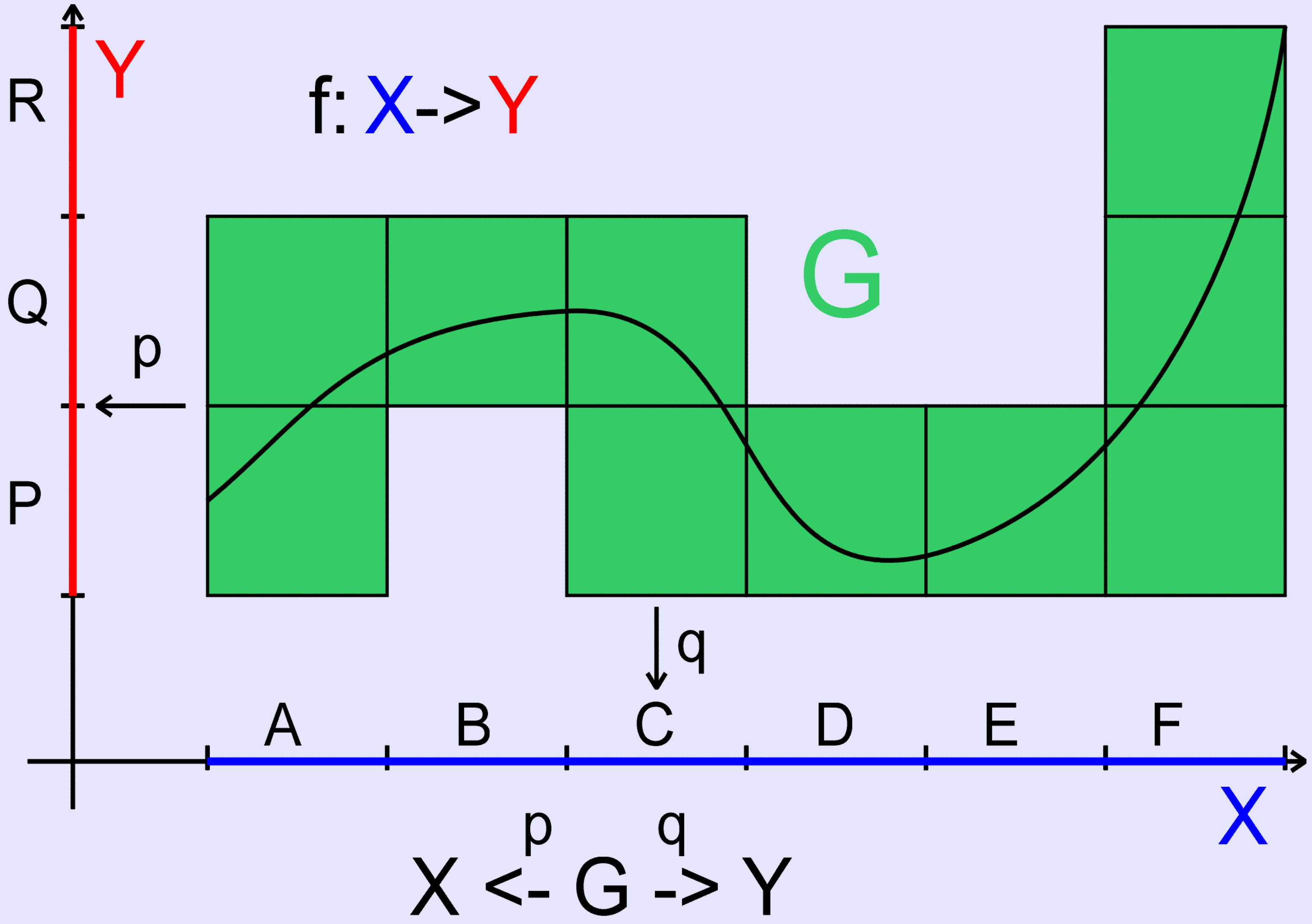
- the diagram (Górniewicz, Granas, 1981) commutes

$$\begin{array}{ccccc} & & G(F) & & \\ & \swarrow p & & \searrow q & \\ X & & \overset{F}{\rightrightarrows} & & Y \end{array}$$

Theorem. (K. Mischaikow, MM, P. Pilarczyk, 2005)
If $F : X \rightrightarrows Y$ is an upper semicontinuous, acyclic valued representation of $f : X \to Y$, then

$$H_*(f) = H_*(q)H_*(p)^{-1}.$$

Graph approach

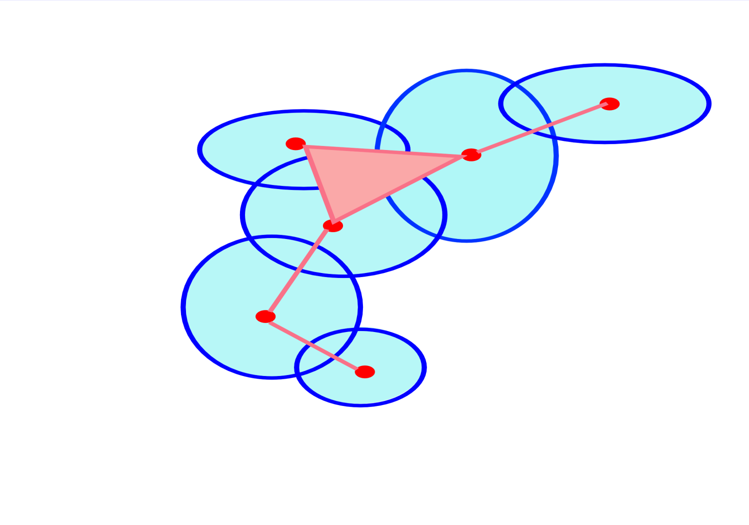


The homology map algorithm

(1) Construct a combinatorial enclosure $\mathcal{F}$ of $f : X \to Y$.
(2) Construct the graph G of $F := \lceil \mathcal{F} \rceil$.
(3) If the homologies of the values of F are not trivial, refine the grid and go to 1.
(4) Find the homologies of the projections $p : G \to X$ and $q : G \to Y$.
(5) Return $H_*(q)H_*(p)^{-1}$

- Pilarczyk (2005) — implementation
- satisfactorily fast for a class of practical problems
- too slow for many other problems

Čech approach

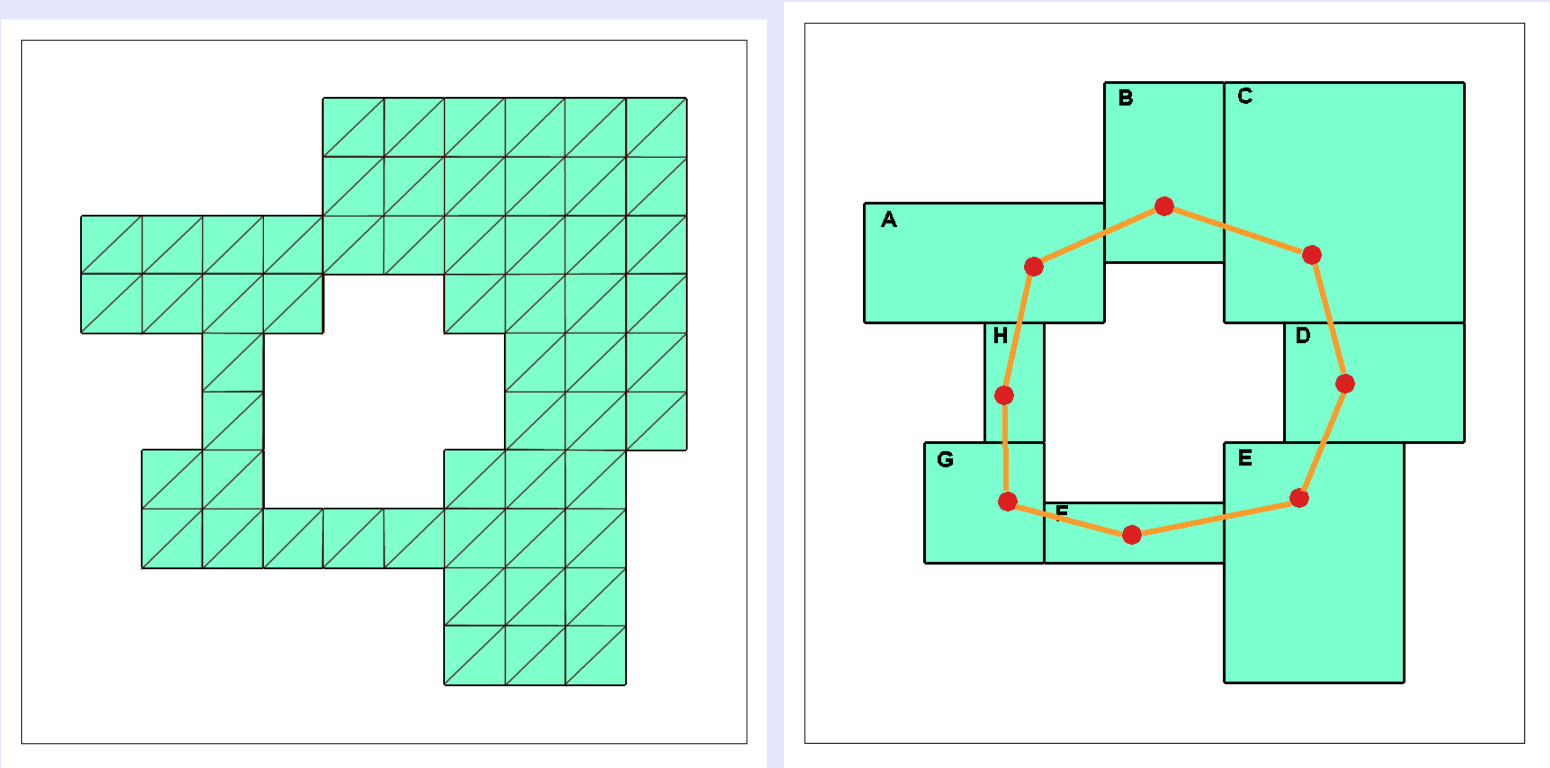


- Čech structure: a finite, non-empty family $\mathcal{X}$ of non-empty, compact, convex subsets of $\mathbb{R}^d$
- support of $\mathcal{X}$: the union of $\mathcal{X}$, denoted $|\mathcal{X}| := \bigcup \mathcal{X}$
- Čech polyhedron: the support of a Čech structure
- nerve of a Čech structure

$K(\mathcal{X})$ has the structure of an abstract simplicial complex.

Theorem. Corollary of the Nerve Theorem (Borsuk 1948, Weil 1952, Wu 1962 ...) The homology of a nerve of a Čech polyhedron X does not depend on the Čech structure and is isomorphic to the singular homology of X.

Čech representation



Simplicial versus Čech representation

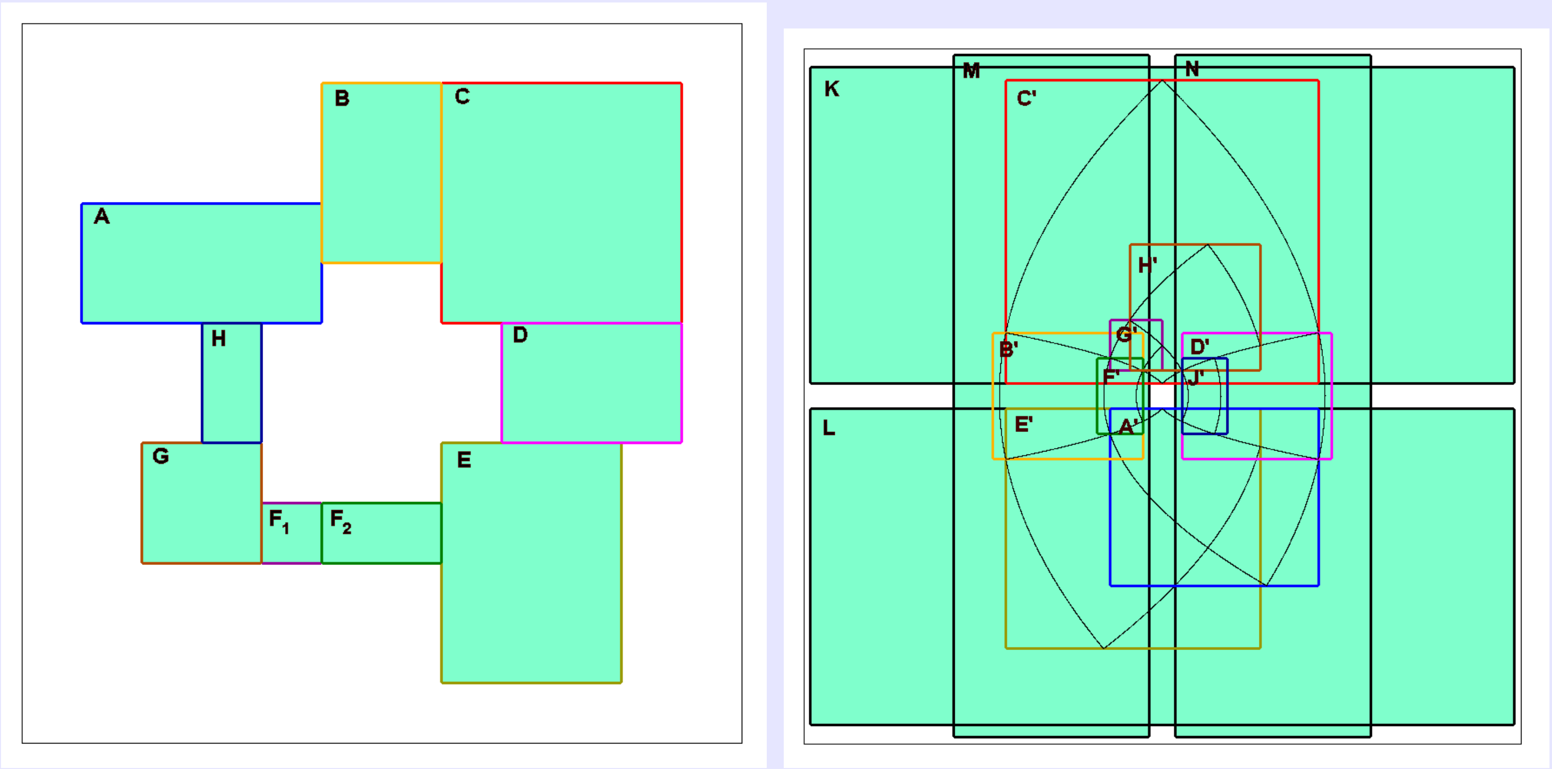
Alternative based on Čech structures

- choose a Čech structure $\mathcal{X}$ on X
- for $Q \in \mathcal{X}$ take $\mathcal{F}(Q)$ as a convex enclosure of $f(Q)$ obtained via rigorous numerics
- $\mathcal{F} : K(\mathcal{X}) \to K(\mathcal{X} \cup \mathcal{F}(\mathcal{X}))$ acts as a simplicial map
- the homology of f is computed when $K(\mathcal{X}) \subset K(\mathcal{X} \cup \mathcal{F}(\mathcal{X}))$ induces an isomorphism
- this is guaranteed when the enclosure is good enough

Example: consider

$$f : \mathbb{C} \ni z \to z^2 \in \mathbb{C}.$$

Alternative based on Čech structures



Visualization of the abstract simplicial complex

Decomposing homology classes on generators

- Let $\mathcal{K}_q(X) = \{ Q_1^q, Q_2^q, \ldots Q_{r_q}^q \}$.
- Let $\{ [u_1], [u_2], \ldots [u_n] \}$ be generators of the homology group $H_q(X)$.
- Let $[z] \in H_q(X)$ be a homology class.
- When computing the homology of maps one faces the problem of finding a representation of $[z]$ in terms of the given generators.

$$[z] = \sum_{i=1}^{n} x_i [u_i]$$

- It reduces to solving the equation

$$z = \sum_{i=1}^{n} x_i u_i + \partial c$$

for some unknown variables $x_1, x_2, \ldots, x_n \in \mathbb{Z}$ and $c \in C_{q+1}(X)$.

Decomposing homology classes on generators

Every chain involved may be represented as a linear combination of elementary chains $\widehat{Q}_j^q$ with respective integer coefficients

$$z = \sum_{j=1}^{r_q} z_j \widehat{Q}_j^q, \quad u_i = \sum_{j=1}^{r_q} u_{ij} \widehat{Q}_j^q, \quad c = \sum_{k=1}^{r_{q+1}} y_k \widehat{Q}_k^{q+1}$$

$$\partial \widehat{Q}_k^{q+1} = \sum_{j=1}^{r_q} a_{kj} \widehat{Q}_j^q$$

$$\partial c = \sum_{j=1}^{r_q} \left(\sum_{k=1}^{r_{q+1}} a_{kj} y_k \right) \widehat{Q}_j^q$$

Therefore our equation reduces to

$$z_j = \sum_{i=1}^{n} u_{ij} x_i + \sum_{k=1}^{r_{q+1}} a_{kj} y_k \text{ for } j = 1, 2, \ldots r_q.$$

which is a system of r_q linear equations with $n + r_{q+1}$ unknowns.

Homology model

- The chain complex $\mathcal{C}^f$ obtained from a reduction algorithm may serve as a convenient model to solve the problems of decomposing homology classes on generators.
- Composing the maps π and ι for the consecutive steps of the reduction algorithm we obtain maps $\pi^f : C \longrightarrow C^f$ and $\iota^f : C^f \longrightarrow C$
- The maps π^f and ι^f may be used to transport homology classes between $H_*(C)$ and $H_*(C^f)$
- Instead of solving the problem in $H_*(C)$ we may transport it to $H_*(C^f)$, solve it there and then transport the solution back to $H_*(C)$

The cost of transporting one generator through π^f or ι^f in general may be quadratic.

In the case of Free Face Reduction Algorithm and Free Coface Reduction Algorithm it is linear!

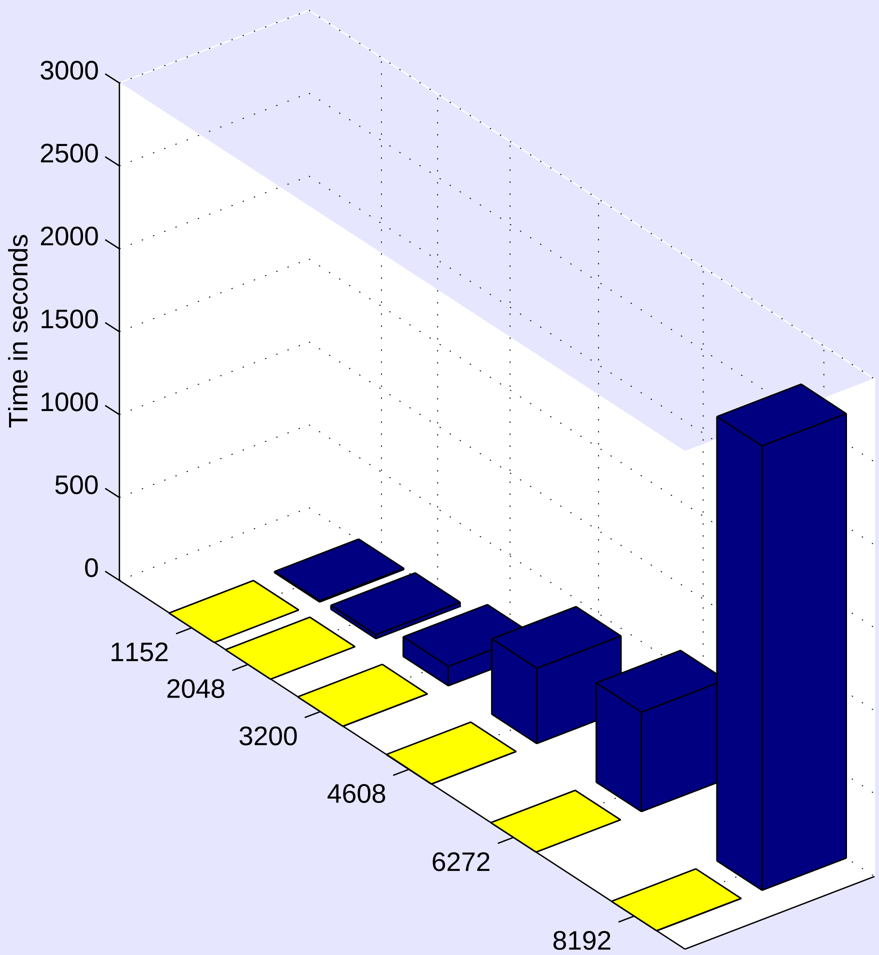
Homology of inclusion maps

- Let $X \subset Y$ be two cubical sets.
- Let $\{ [u_1], [u_2], \ldots [u_m] \}$ and $\{ [w_1], [w_2], \ldots [w_n] \}$ be bases of $H_q(X)$ and $H_q(Y)$ respectively.
- The problem of finding the matrix of the homology of the inclusion map in these bases reduces to decomposing each generator $[u_i]$ of $H_q(X)$ on the basis $\{ [w_1], [w_2], \ldots [w_n] \}$ in $H_q(Y)$.
- Using the diagram

$$\begin{tikzcd} H_*(X) \arrow[r, "i_*"] & H_*(Y) \arrow[d, shift right, "\pi_*^f"'] \\ & H_*(\mathcal{C}^f(Y)) \arrow[u, shift right, "\iota_*^f"'] \end{tikzcd}$$

  we can solve the problem in the homology model $H_*(\mathcal{C}^f(Y))$, where it is much simpler.

PP and CR comparison on an inclusion map



PP and CR comparison on an inclusion map

Size	1152	2048	3200	4608	6272	8192
PP	6.719	24.562	117.625	453.563	600.14	2679.98
CR	0.047	0.063	0.109	0.141	0.187	0.235

Algorithm	Complexity exponent
PP	3.00046
CR	0.849867

Persistent Homology

- Let $X = (X_1, X_2, \ldots, X_n)$ be a filtration of cubical sets.
- Following A. Zomorodian and G. Carlsson for $i < j$ we define the (i, j)-persistent qth homology group of X_i by

$$PH_q^{i,j}(X) := Z_q(X_i)/B_q(X_j) \cap Z_q(X_i)$$

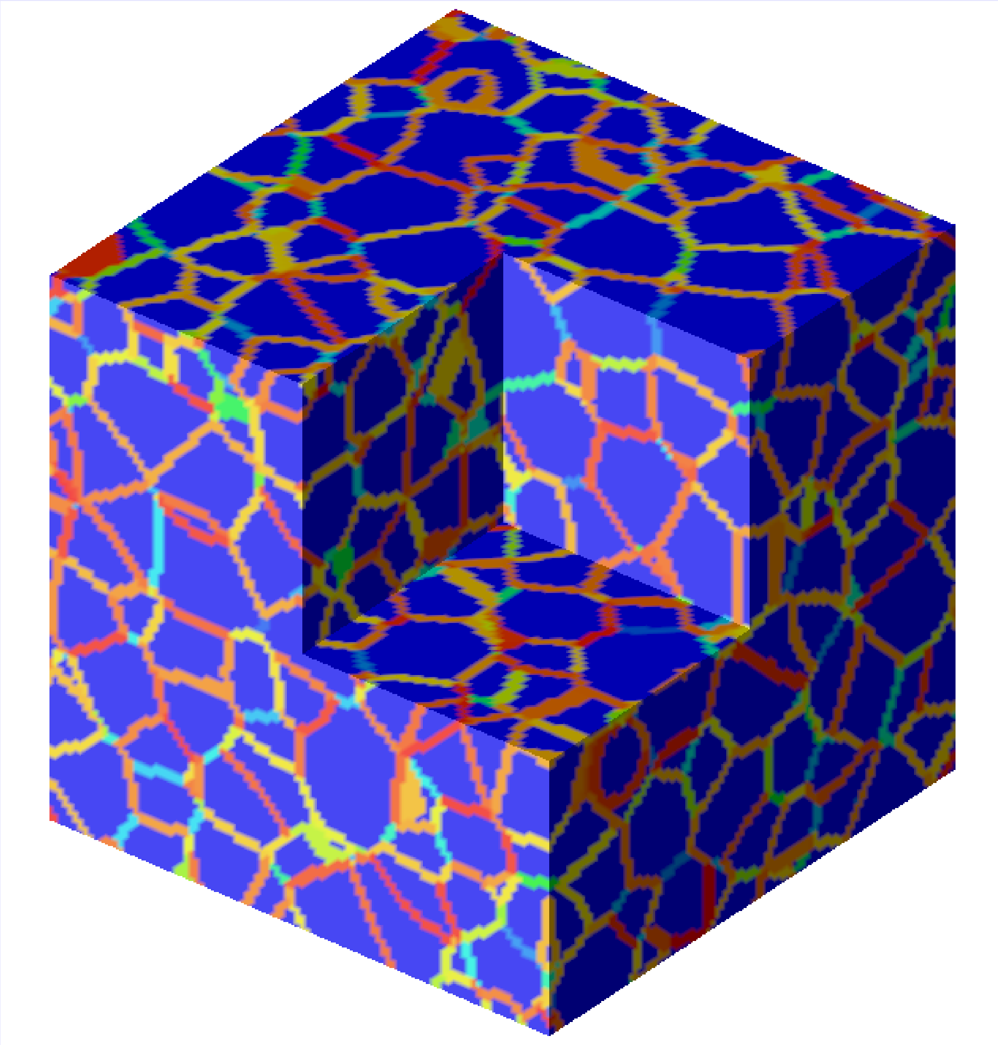
Theorem. (Zomorodian, 2001) Let

$$\iota^{i,j} : X_i \to X_j$$

denote the inclusion map. Then

$$PH_q^{i,j}(X) \cong \operatorname{im} H_q(\iota^{i,j}).$$

Persistent Homology



An example provided by Thomas Wanner.

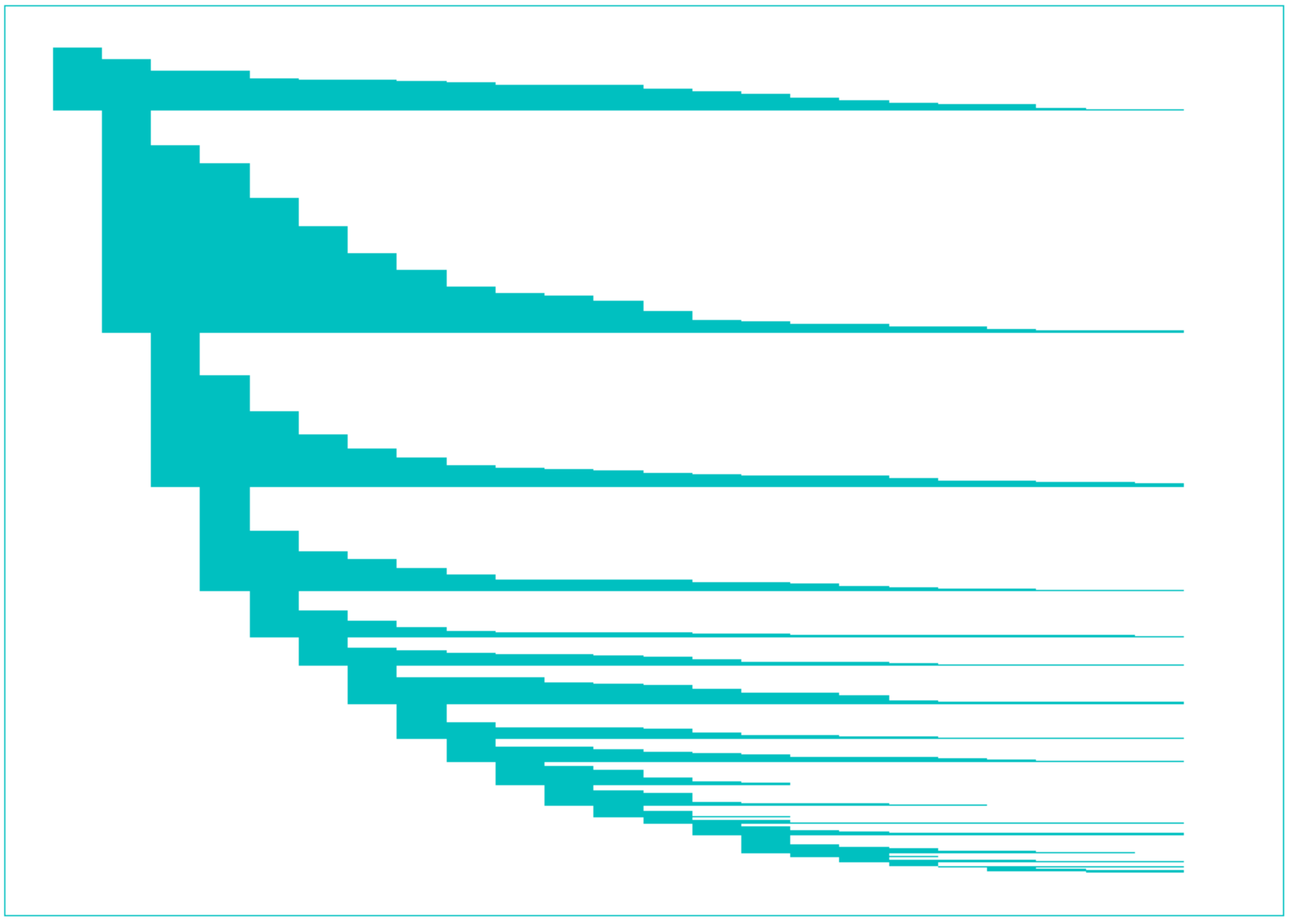
Persistent Homology

The ranks of the persistent homology groups in the example are collected in the table

$PH_1^{i,j}$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	59	47	37	37	30	28	28	27	26	24	24	23	20	18	15	11	9	7	5	5	2	1	1
2	-	256	213	196	156	128	103	86	70	61	59	53	40	30	26	20	17	13	11	9	4	3	3
3	-	-	357	300	227	177	139	113	90	79	75	68	53	42	37	30	27	21	17	15	9	7	6
4	-	-	-	398	284	214	169	135	106	90	86	79	64	50	45	37	32	25	20	17	10	8	7
5	-	-	-	-	327	239	184	144	112	95	91	84	69	53	48	39	34	27	22	19	12	10	8
6	-	-	-	-	-	266	201	159	124	106	102	94	78	59	52	43	38	30	24	20	13	11	9
7	-	-	-	-	-	-	237	185	149	131	123	113	96	73	63	54	46	34	27	23	16	14	12
8	-	-	-	-	-	-	-	217	164	142	134	123	105	79	66	57	48	36	28	24	17	15	13
9	-	-	-	-	-	-	-	-	186	156	148	135	114	87	73	62	53	41	32	26	18	16	14
10	-	-	-	-	-	-	-	-	-	177	165	149	120	90	75	62	53	41	32	26	18	16	14
11	-	-	-	-	-	-	-	-	-	-	185	163	132	94	78	64	55	42	33	26	18	16	14
12	-	-	-	-	-	-	-	-	-	-	-	174	138	95	79	64	55	42	33	26	18	16	14
13	-	-	-	-	-	-	-	-	-	-	-	-	144	99	83	65	56	43	34	27	19	17	15
14	-	-	-	-	-	-	-	-	-	-	-	-	-	109	91	69	59	45	36	29	21	19	17
15	-	-	-	-	-	-	-	-	-	-	-	-	-	-	108	77	65	49	38	31	22	20	17
16	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	81	68	50	38	31	22	20	17
17	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	73	53	41	33	23	21	18
18	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	56	41	33	23	21	18
19	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	43	34	24	22	19
20	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	37	26	23	20
21	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	26	23	20
22	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	25	21
23	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	22

Total computation time is below 3h on Intel Core 2 Duo 2GHz processor and 2GB RAM

Persistent Homology



References

- **A. Granas and L. Górniewicz**, Some general theorems in coincidence theory, *J. Math. Pure Appl.* (**1981**).
- **M. Allili and T. Kaczynski**, An algorithmic approach to the construction of homomorphisms induced by maps in homology, *Trans. Amer. Math. Soc.* (**2000**).
- **T. Kaczynski, K. Mischaikow, MM**, Computational Homology, *Springer Verlag* (**2004**).
- **K. Mischaikow, MM, P. Pilarczyk**, Graph Approach to the Computation of the Homology of Continuous Maps, *Foundations of Computational Mathematics* (**2005**).
- **M. Mazur, MM, J. Szybowski**, Algebraic construction of a coboundary of a given cycle, *Opuscula Mathematica* (**2007**).
- **MM**, Čech Type Approach to Computing Homology of Maps, preprint.
- **MM, Th. Wanner**, Coreduction homology algorithm for inclusions and persistent homology, in preparation.