Computational homology in dynamical systems

Marian Mrozek

Jagiellonian University, Kraków

National University of Ireland, Galway
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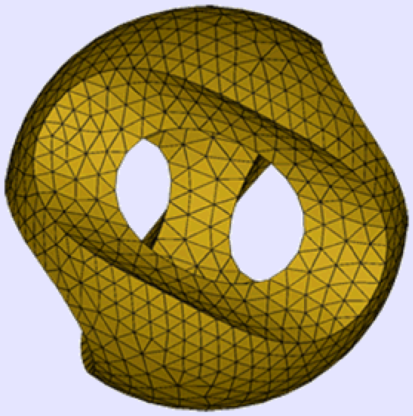
Outline

- Dynamical systems
- Rigorous numerics of dynamical systems
- Homological invariants of dynamical systems
- **Computing homological invariants**
- Homology algorithms for subsets of $\mathbb{R}^d$
- Homology algorithms for maps of subsets of $\mathbb{R}^d$
- Applications

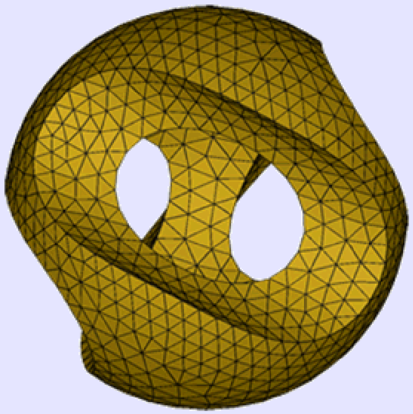
Discretization of space $_3$

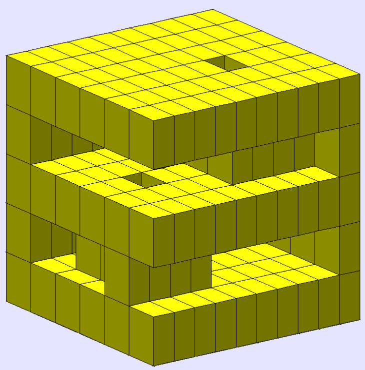


Simplicial complex

- standard

Discretization of space





Simplicial complex

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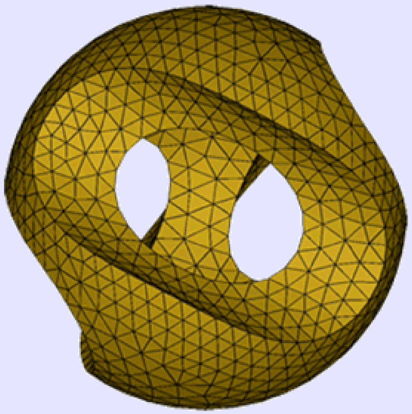
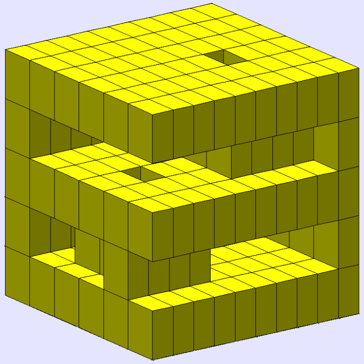
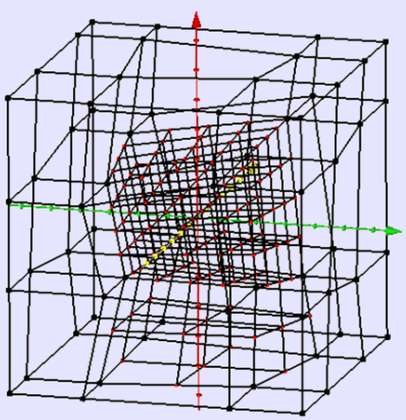
Cubical set

- typical in rigorous numerics and imaging
- very efficient and fast representation (bitmaps)

Discretization of space







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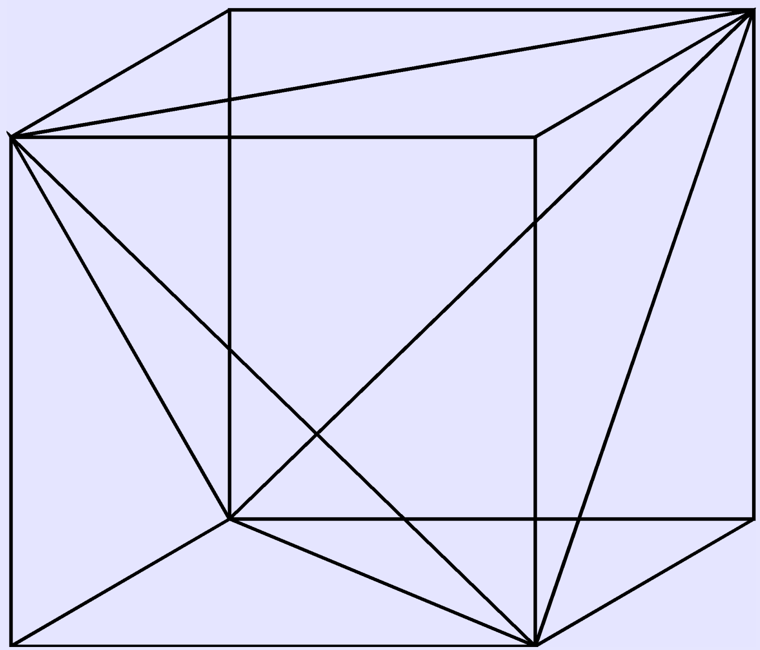
General polyhedrons

- most general
- obtaining the chain complex is not straightforward
- not convenient in implementation

Cube triangulation

- How many simplices do we need to triangulate a d-cube?
- Not more than $d!$ but can we do better?



- $C(d)$ - cover number
- $T(d)$ - triangulation number
- $T^v(d)$ - vertex triangulation number

$$C(d) \leq T(d) \leq T^v(d)$$

Cube triangulation

Theorem. Hughes, Anderson (1995), Bliss, Su (2005)

d	1	2	3	4	5	6	7
$T^v(d)$	1	2	5	16	67	308	1493
$T(d)$	1	2	5	16	?	?	?

Theorem. Smith (2000)

$$C(d) \geq \frac{6^{d/2} d!}{2(d+1)^{(d+1)/2}}$$

Elementary intervals and cubes

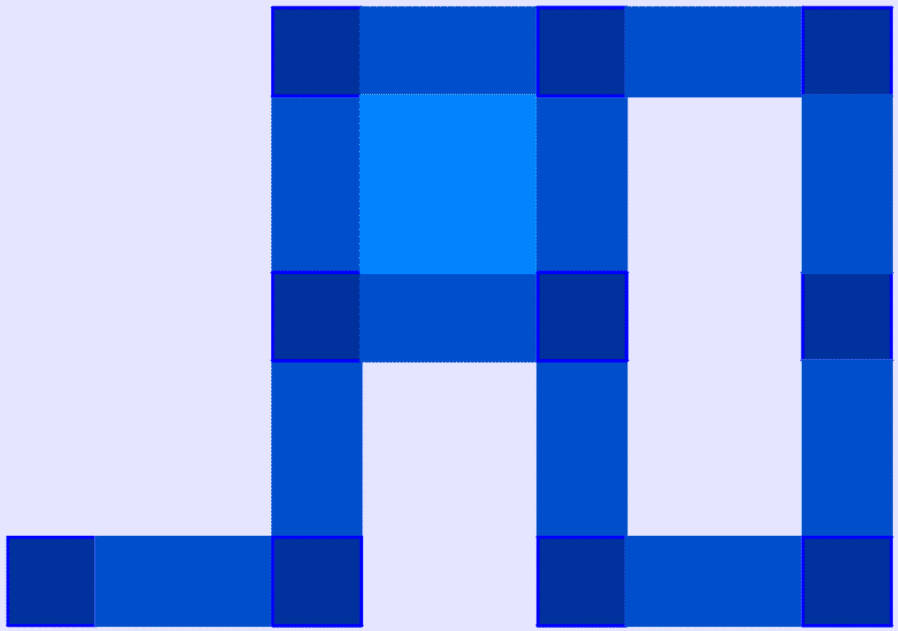
- An elementary interval is an interval $[k, l] \subset \mathbb{R}$ such that $l = k$ (degenerate) or $l = k + 1$ (nondegenerate).
- An elementary cube Q in $\mathbb{R}^d$ is
$$I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d.$$
- The dimension of Q is the number of nondegenerate I_i.
- $\mathcal{K}$ — the set of all elementary cubes in $\mathbb{R}^d$
- $\mathcal{K}_k$ — the set of all elementary cubes in $\mathbb{R}^d$ of dimension k
- An elementary cube is full if its dimension is d.
- For $\mathcal{A} \subset \mathcal{K}$ we use notation $|\mathcal{A}| := \bigcup \mathcal{A}$.
- For $A \subset \mathbb{R}^d$ we use notation $\mathcal{K}(A) := \{\, Q \in \mathcal{K} \mid Q \subset A \,\}$.

Cubical sets

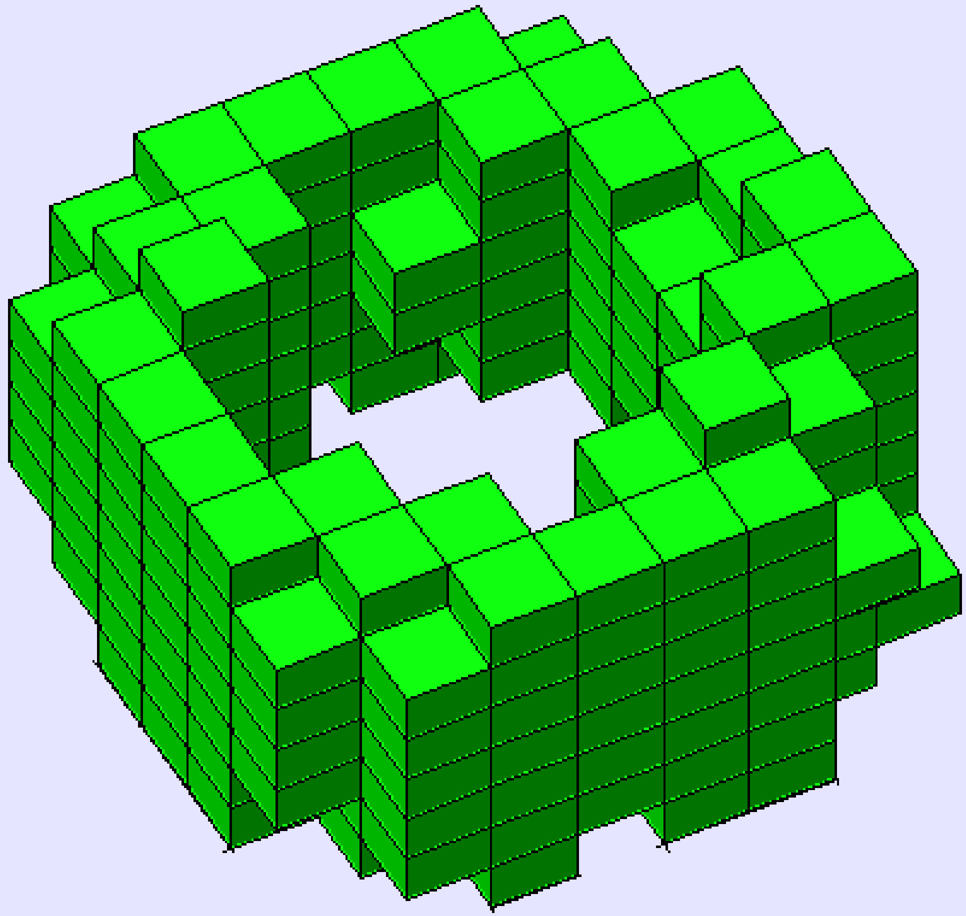
- The set $A \subset \mathbb{R}^d$ is cubical if there exists a finite family $\mathcal{A} \subset \mathcal{K}$ such that $A = |\mathcal{A}|$.
- The family $\mathcal{A}$ is referred to as the representation of A.
- The unique minimal representation, the minimal representation of A, is denoted by $\mathcal{K}_{\mathsf{min}}(A)$.
- A cubical set is a full cubical set if its minimal representation consists only of full elementary cubes.

Theorem. (Blass, Holsztyński, 1972) Every polyhedron is homeomorphic to a cubical set.

A cubical set in $\mathbb{R}^2$



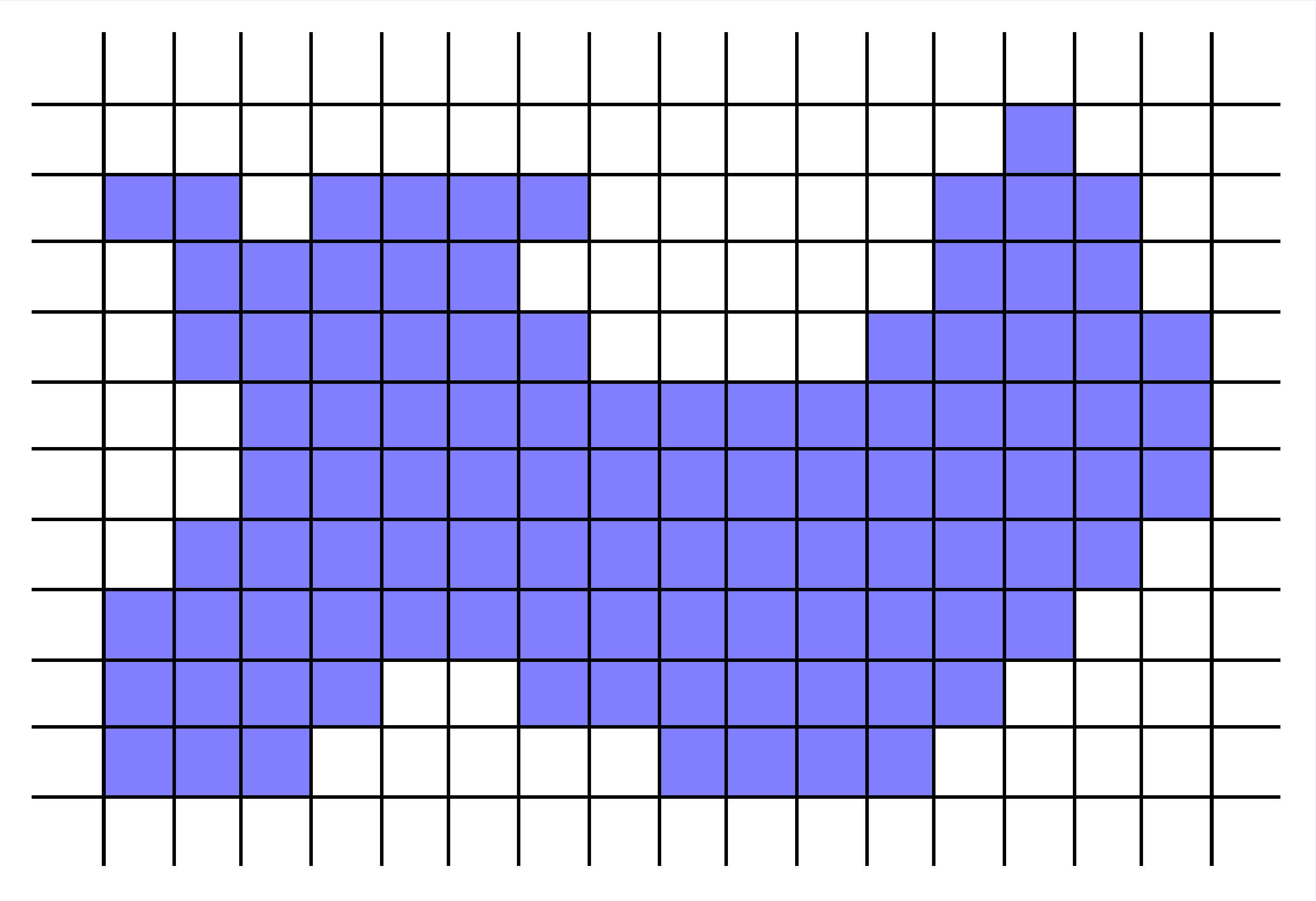
A full cubical set in $\mathbb{R}^3$



Combinatorial boundary and interior

- For $A \subset \mathbb{R}^d$ define
$$o_d(A) := \{\, Q \in \mathcal{K}_d \mid Q \cap A \neq \emptyset \,\},$$
- For $\mathcal{N} \subset \mathcal{K}_d$ define
$$\operatorname{int} \mathcal{N} := \{\, Q \in \mathcal{N} \mid o_d(Q) \subset \mathcal{N} \,\},$$
$$\operatorname{bd} \mathcal{N} := \mathcal{N} \setminus \operatorname{int}(\mathcal{N}).$$

Combinatorial boundary and interior



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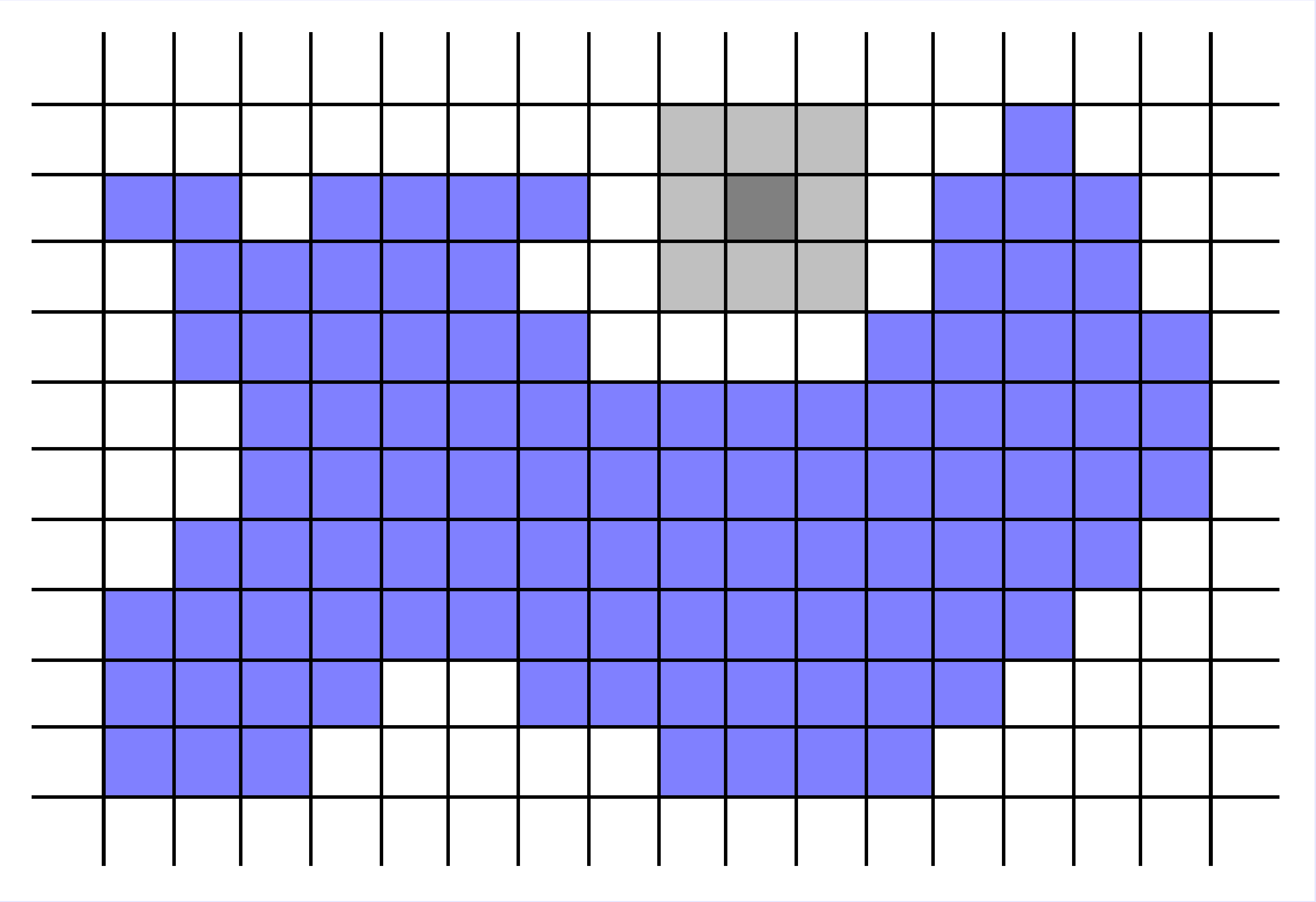

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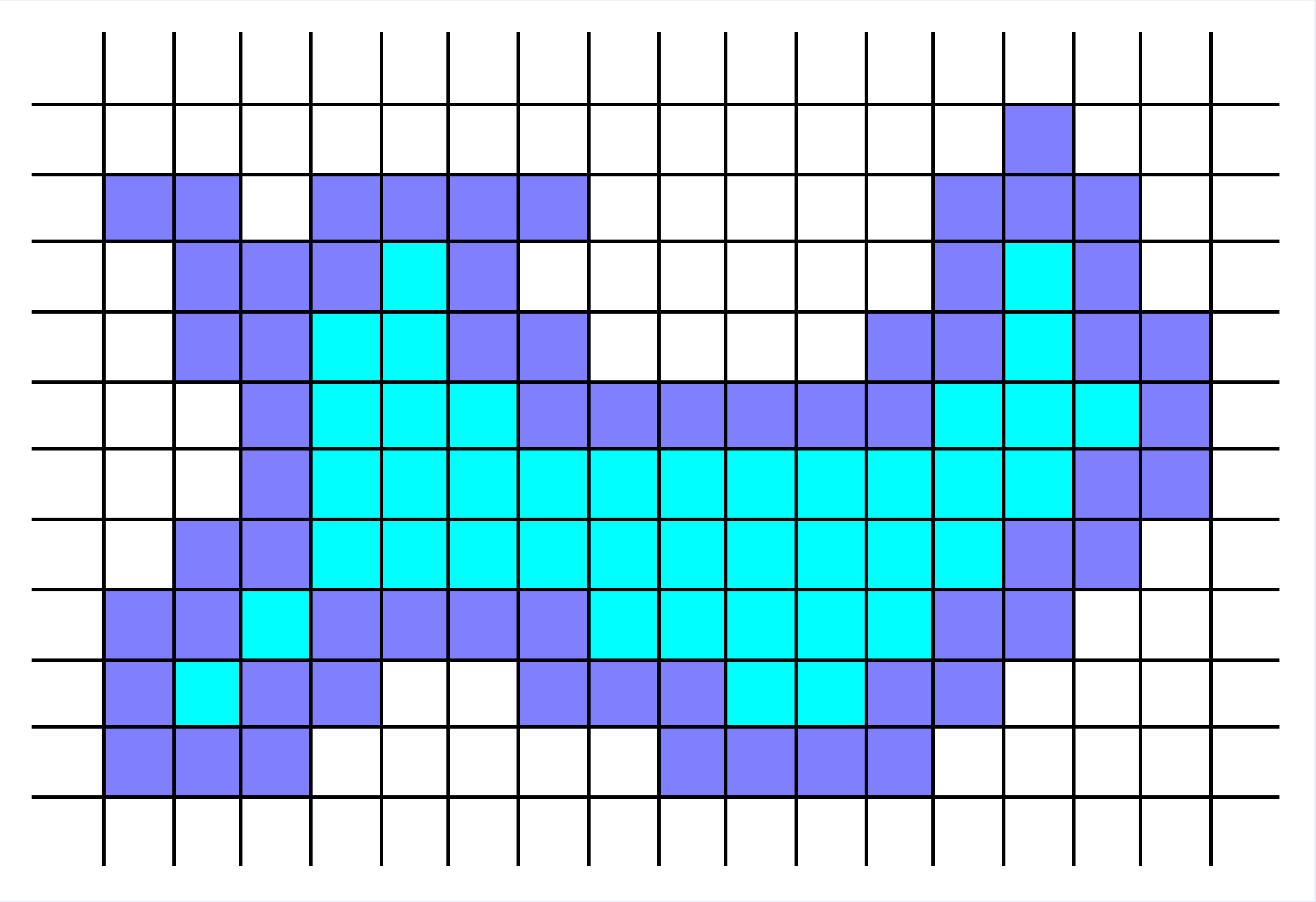
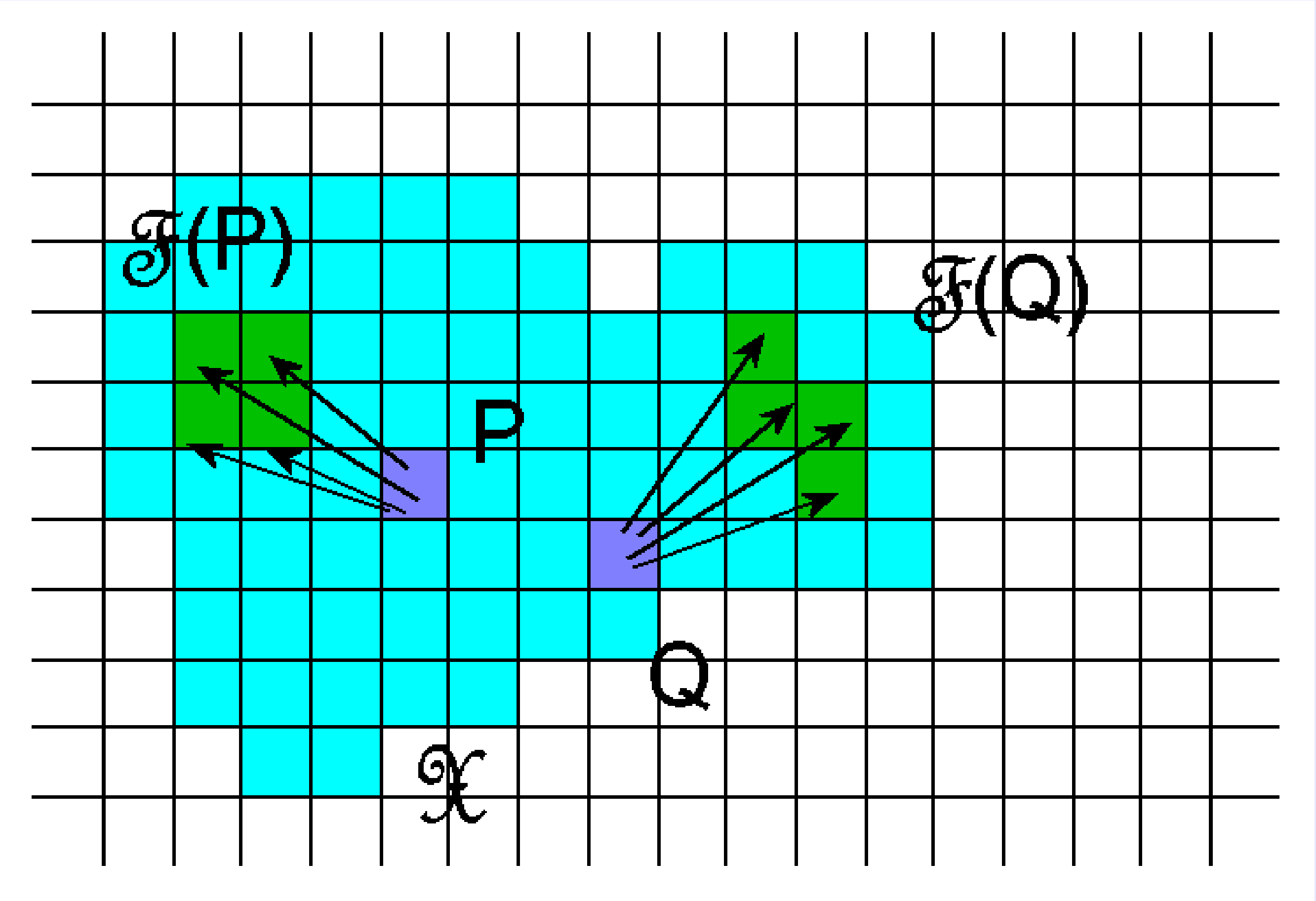


Multivalued combinatorial maps

- $\mathcal{X} \subset \mathcal{K}^d$ — a finite subfamily
- $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ — a multivalued combinatorial map
- The associated digraph has $\mathcal{X}$ as the set of vertices and an edge from P to Q iff $Q \in \mathcal{F}(P)$.

Combinatorial boundary and interior



Combinatorial enclosures

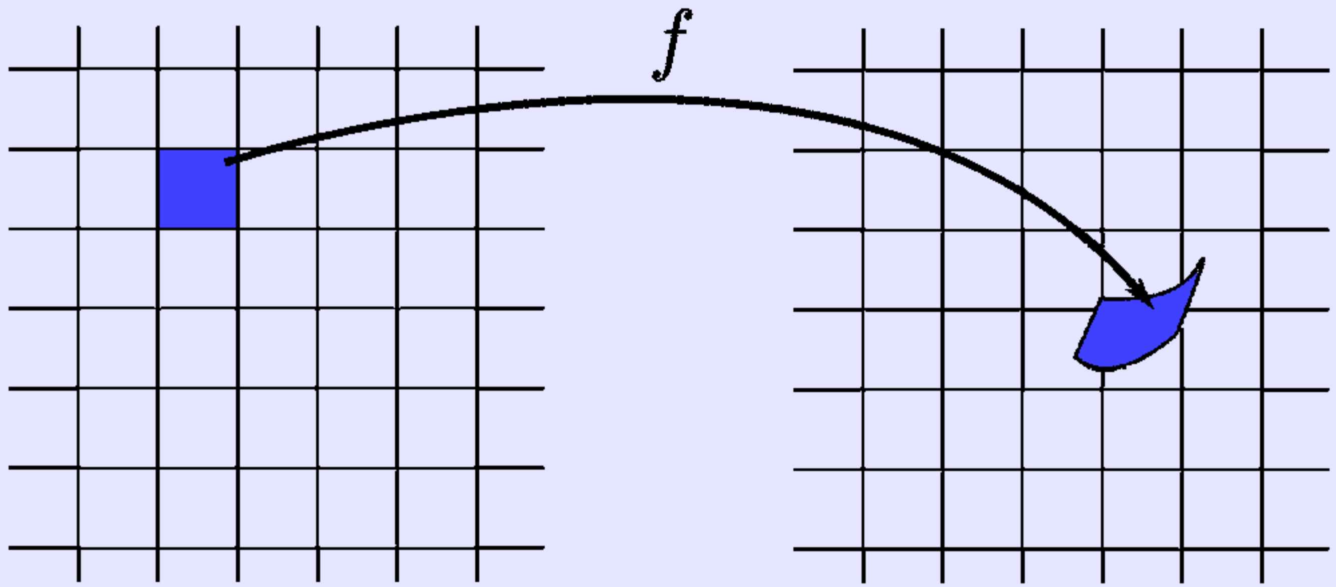
- A combinatorial multivalued map $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ is a combinatorial enclosure of $f : X \to X$ if for every $Q \in \mathcal{X}$

$$o_d(f(Q)) \subset \mathcal{F}(Q).$$

- In this case we say that f is a selector of $\mathcal{F}$.
- If $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ is a combinatorial enclosure of $f : X \to X$, then for every $Q \in \mathcal{X}$

$$f(Q) \subset \operatorname{int} |\mathcal{F}(Q)|.$$

Combinatorial enclosures



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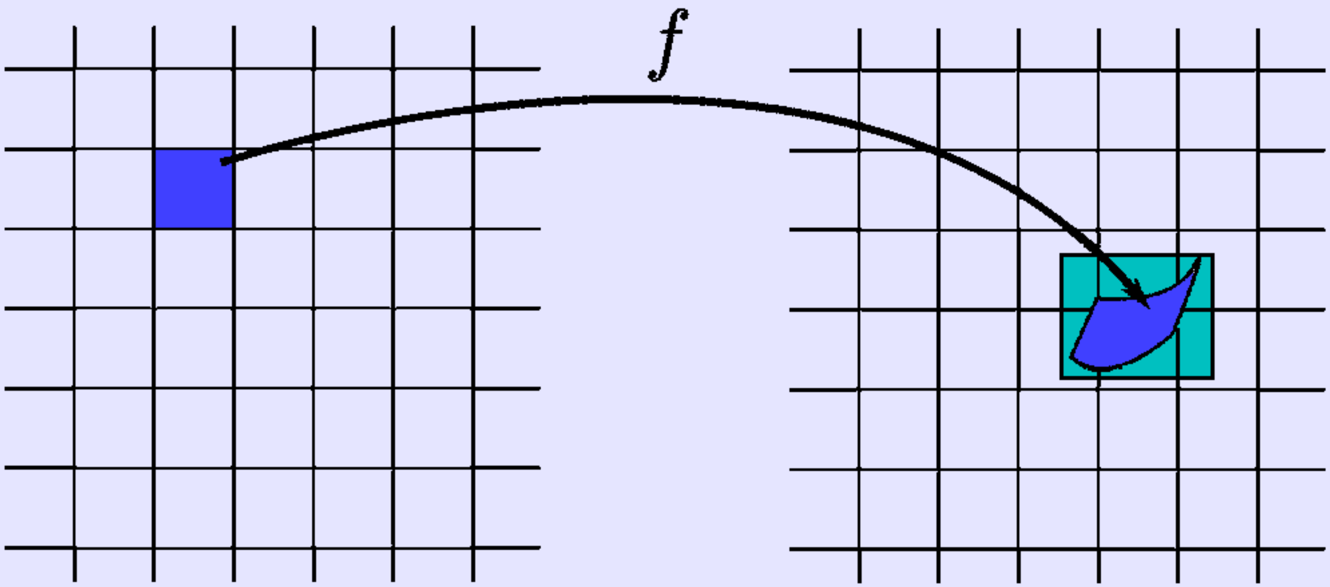

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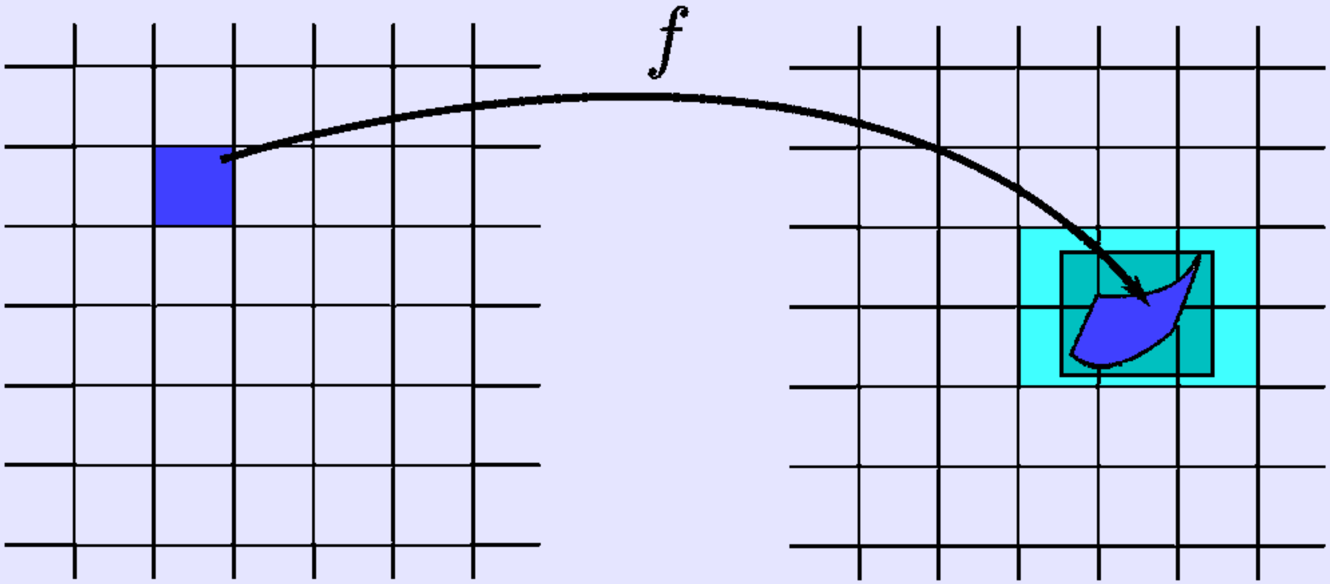
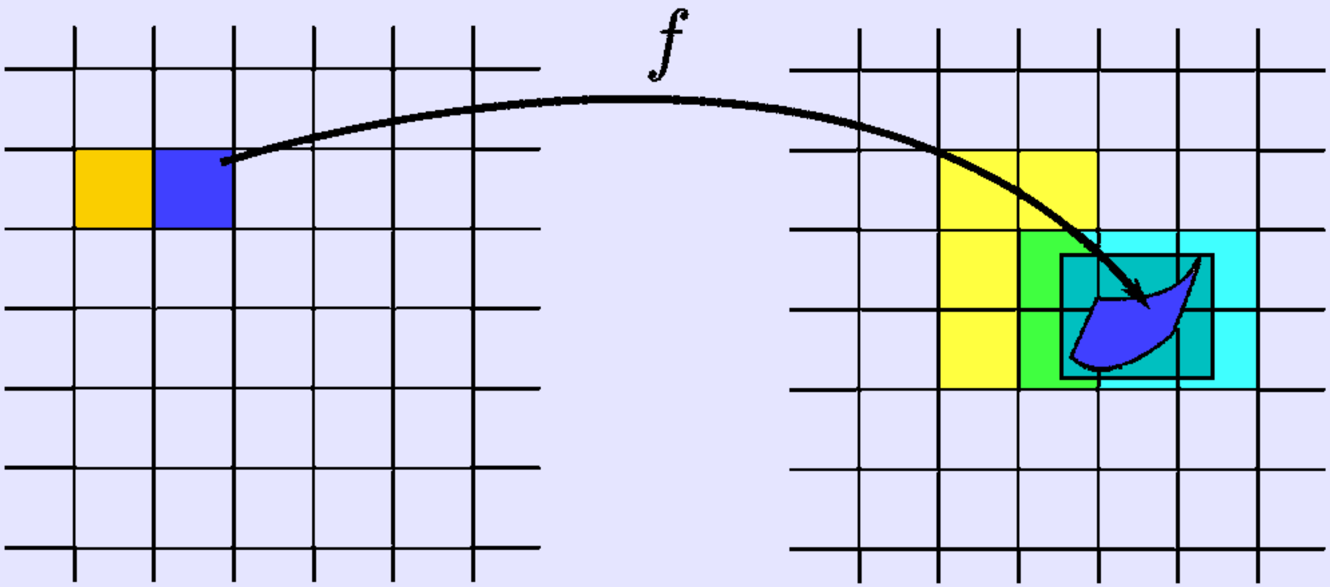
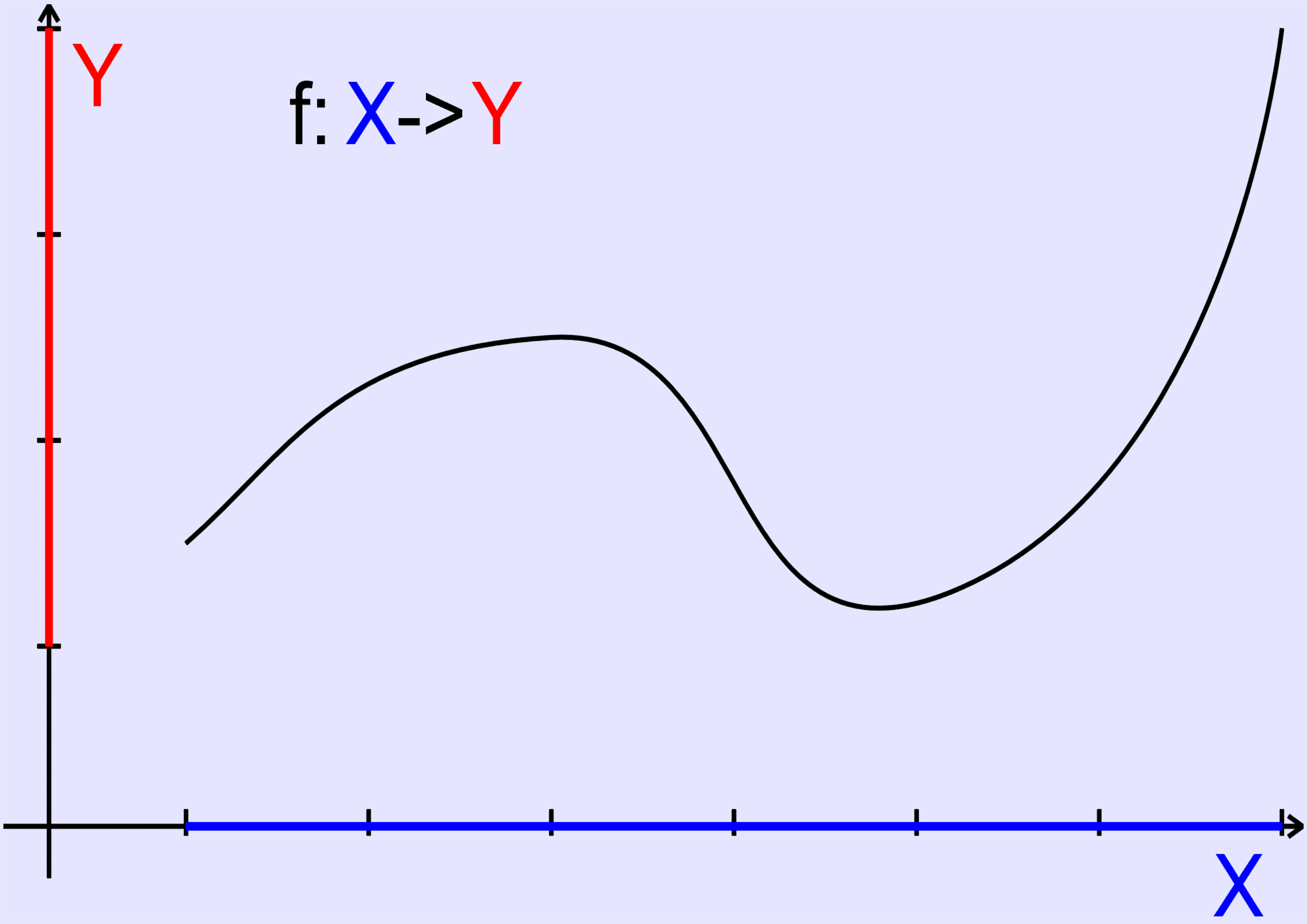


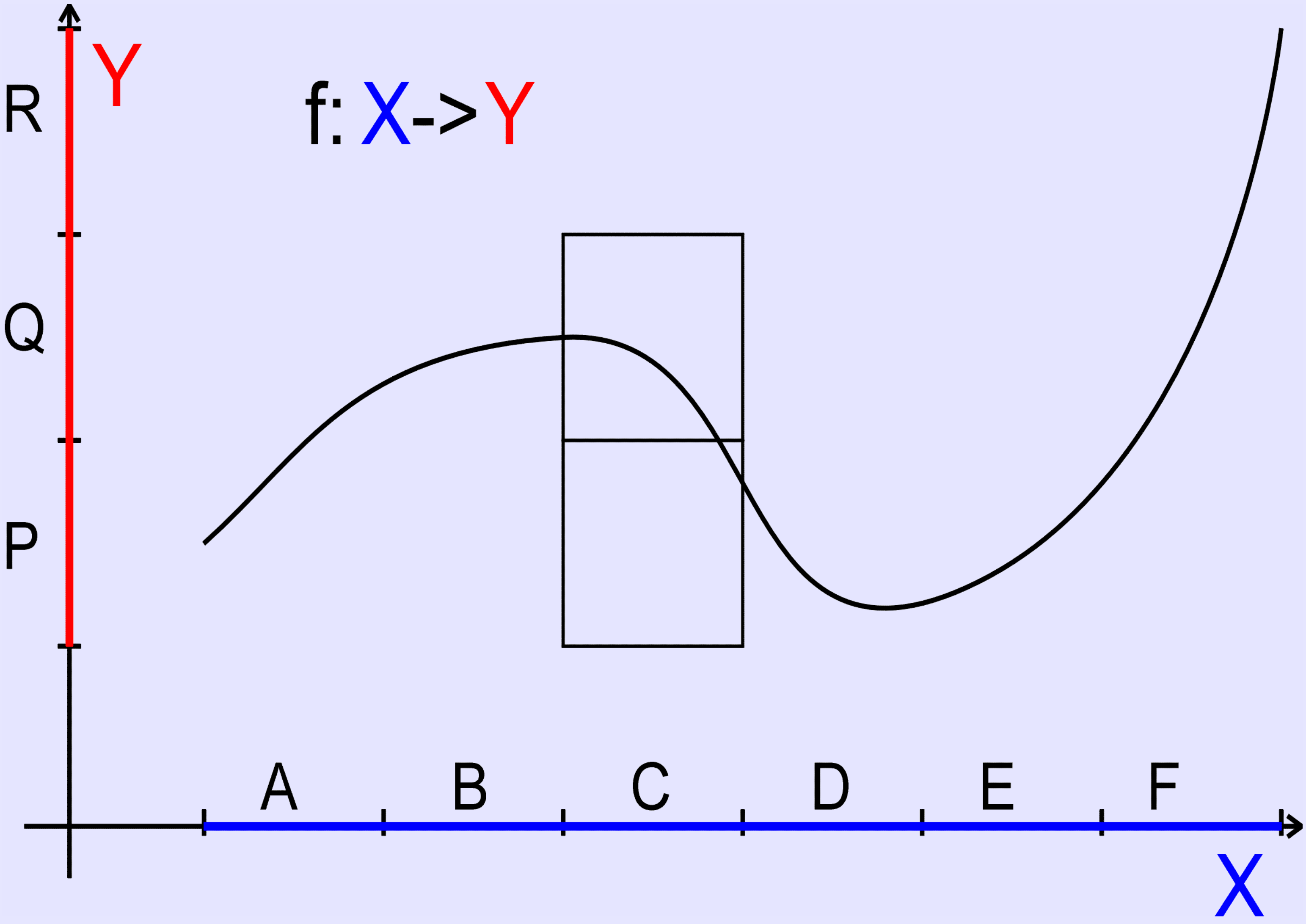
Combinatorial enclosures



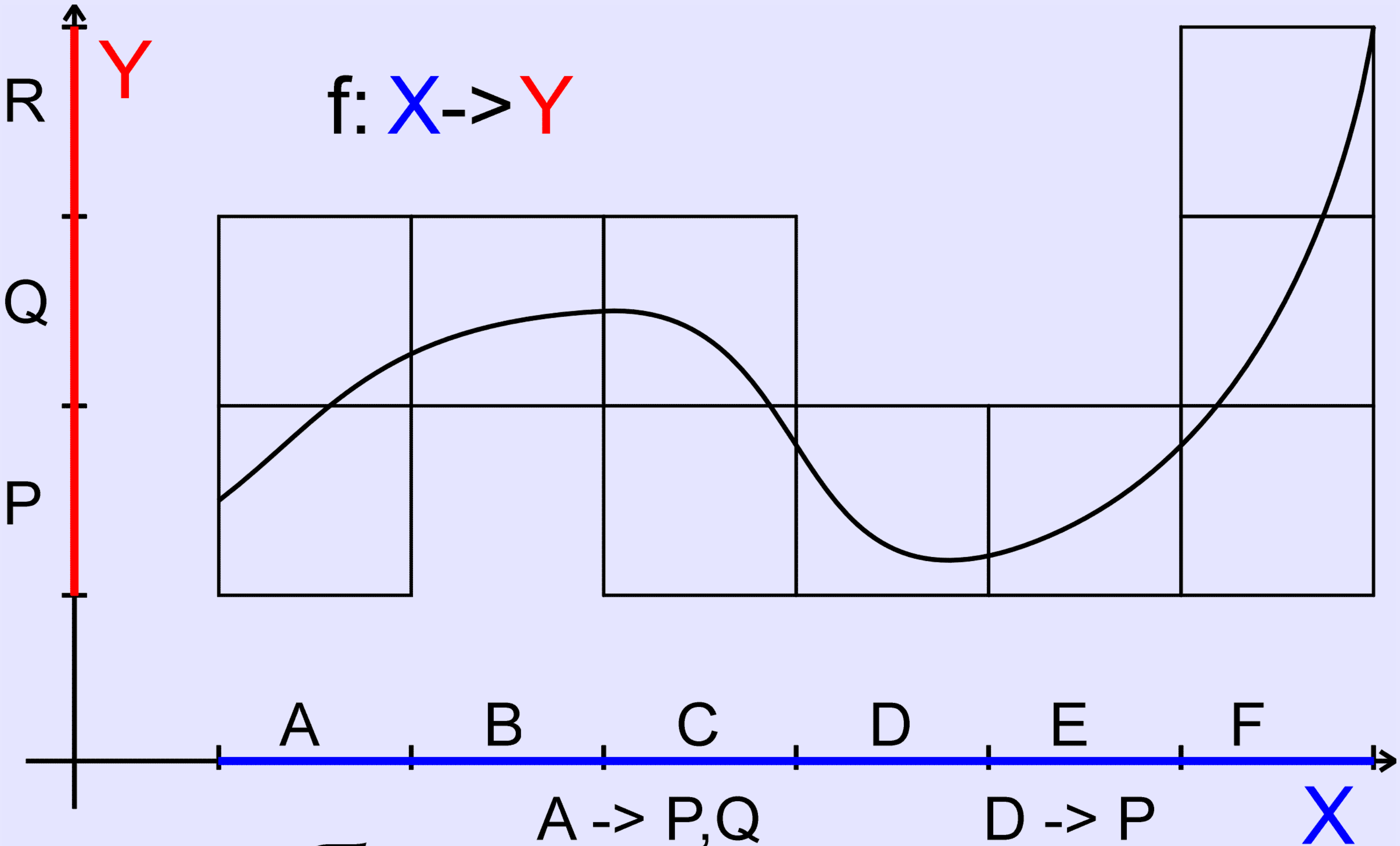
Graph of a continuous map f



Estimates of values on the grid of cubes



Multivalued representation $\mathcal{F}$



$\mathcal{F}$:

A -> P,Q
B -> Q
C -> P,Q
D -> P
E -> P
F -> P,Q,R

Combinatorial solutions

- Let I be an interval in $\mathbb{Z}$ containing 0.
- A solution through $Q \in \mathcal{K}$ under $\mathcal{F}$ is a function $\Gamma : I \to \mathcal{K}$ satisfying the following two properties:

  (1) $\Gamma(0) = Q$,

  (2) $\Gamma(n+1) \in \mathcal{F}(\Gamma(n))$ for all n such that $n, n+1 \in I$.
- In the language of the associated digraph a solution is just a path in the digraph

Combinatorial invariant parts

Assume $\mathcal{N} \subset \mathcal{K}$ is finite. The invariant part of $\mathcal{N}$ under $\mathcal{F}$ is

$$\mathrm{Inv}(\mathcal{N},\mathcal{F}) := \{\, Q \in \mathcal{N} \mid \text{there exists a full solution } \Gamma : \mathbb{Z} \to \mathcal{N} \,\}.$$

The positively invariant part and the negatively invariant part of $\mathcal{N}$ under $\mathcal{F}$ are defined respectively by

$$\begin{aligned}\mathrm{Inv}^{+}(\mathcal{N},\mathcal{F}) &:= \{\, Q \in \mathcal{N} \mid \text{there exists a solution } \Gamma : \mathbb{Z}^{+} \to \mathcal{N} \,\}\\ \mathrm{Inv}^{-}(\mathcal{N},\mathcal{F}) &:= \{\, Q \in \mathcal{N} \mid \text{there exists a solution } \Gamma : \mathbb{Z}^{-} \to \mathcal{N} \,\}\end{aligned}$$

We have the following obvious formula

$$\mathrm{Inv}(\mathcal{N},\mathcal{F}) = \mathrm{Inv}^{-}(\mathcal{N},\mathcal{F}) \cap \mathrm{Inv}^{+}(\mathcal{N},\mathcal{F}).$$

Algorithmizable formulae for invariant parts

Let $\mathcal{F}_{\mathcal{N}} : \mathcal{N} \rightrightarrows \mathcal{N}$ denote the map given by

$$\mathcal{F}_{\mathcal{N}}(Q) := \mathcal{F}(Q) \cap \mathcal{N}.$$

There exists an integer n such that

$$\mathrm{Inv}^{+}(\mathcal{N}, \mathcal{F}) = \bigcap_{i=0}^{n} \mathcal{F}_{\mathcal{N}}^{i}(\mathcal{N})$$

$$\mathrm{Inv}^{-}(\mathcal{N}, \mathcal{F}) = \bigcap_{i=0}^{n} \mathcal{F}_{\mathcal{N}}^{-i}(\mathcal{N})$$

Combinatorial Index Pairs.

A finite subset $\mathcal{N}$ of $\mathcal{K}_d$ is an isolating neighborhood for $\mathcal{F}$ if

$$\mathrm{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathrm{int}\,\mathcal{N}.$$

We say that $(\mathcal{P}_1, \mathcal{P}_2)$ is a combinatorial index pair for $\mathcal{F}$ in $\mathcal{N}$ if $\mathcal{P}_2 \subset \mathcal{P}_1 \subset \mathcal{N}$ and the following three conditions are satisfied.

- (positive relative invariance)

$$\mathcal{F}(\mathcal{P}_i) \cap \mathcal{N} \subset \mathcal{P}_i$$

- (exit set)

$$\mathcal{F}(\mathcal{P}_1) \cap \mathrm{bd}\,\mathcal{N} \subset \mathcal{P}_2$$

- (isolation)

$$\mathrm{Inv}(\mathcal{N}, \mathcal{F}) \subset \mathcal{P}_1 \setminus \mathcal{P}_2$$

Index Pairs from Combinatorial Index Pairs.

Theorem. (A. Szymczak 1997, MM 1996,2006)
Assume $\mathcal{N}$ is an isolating neighborhood for $\mathcal{F}$ and $(\mathcal{P}_1, \mathcal{P}_2)$ is a combinatorial index pair for $\mathcal{F}$ in $\mathcal{N}$. Then for any selector f of $\mathcal{F}$ the set $|\mathcal{N}|$ is an isolating neighborhood for f and $(|\mathcal{P}_1|, |\mathcal{P}_2|)$ is a index pair for f.

Construction of index quadruples

Theorem. (MM,2005)
Assume $\mathcal{N}$ is an isolating neighborhood for $\mathcal{F}$. Let

$$\begin{aligned}\mathcal{P}_1 &:= \operatorname{Inv}^-(\mathcal{N},\mathcal{F}),\\ \mathcal{P}_2 &:= \operatorname{Inv}^-(\mathcal{N},\mathcal{F}) \setminus \operatorname{Inv}^+(\mathcal{N},\mathcal{F}).\end{aligned}$$

Then $(\mathcal{P}_1, \mathcal{P}_2)$ is a combinatorial index pair for $\mathcal{F}$ in $\mathcal{N}$ and

$$|\mathcal{P}_1| \setminus |\mathcal{P}_2| \subset \operatorname{int} |\mathcal{N}|.$$

Moreover, if

$$\begin{aligned}\bar{\mathcal{P}}_1 &:= \mathcal{P}_1 \cup \mathcal{F}(\mathcal{P}_1),\\ \bar{\mathcal{P}}_2 &:= \mathcal{P}_2 \cup (\mathcal{F}(\mathcal{P}_1) \setminus \mathcal{P}_1),\end{aligned}$$

then for any selector f of $\mathcal{F}$ the quadruple $(|\mathcal{P}_1|, |\mathcal{P}_1|, |\bar{\mathcal{P}}_1|, |\bar{\mathcal{P}}_2|)$ is an index quadruple.

Positive invariant part algorithm

```
function positiveInvariantPart(set N, combinatorialMap F)
F := restrictedMap(F, N);
S := C := N;
repeat
  S' = S;
  C := evaluate(F, C);
  S := S ∩ C;
until (S = S');
return S;
```

Proposition. Assume the algorithm is called with a collection of cubes $\mathcal{N}$ and a combinatorial multivalued map $\mathcal{F}$ on input. Then it always stops and returns the positive invariant part of $\mathcal{F}$ in $\mathcal{N}$.

Combinatorial Index Pair Algorithm

function $\texttt{combinatorialIndexPair}(\texttt{set N}, \texttt{combinatorialMap F})$
$\texttt{S}^+ := \texttt{positiveInvariantPart}(\texttt{N}, \texttt{F});$
$\texttt{Finv} := \texttt{evaluateInverse}(\texttt{F});$
$\texttt{S}^- := \texttt{positiveInvariantPart}(\texttt{N}, \texttt{Finv});$
if $\texttt{S}^- \cap \texttt{S}^+ \subset \texttt{int}(\texttt{N})$ **then**
  $\texttt{P}_1 := \texttt{S}^-;$
  $\texttt{P}_2 := \texttt{S}^- \setminus \texttt{S}^+;$
  $\bar{\texttt{P}}_1 := \texttt{P}_1 \cup \texttt{F}(\texttt{P}_1);$
  $\bar{\texttt{P}}_2 := \texttt{P}_2 \cup \texttt{F}(\texttt{P}_1) \setminus \texttt{P}_1;$
  return $(\texttt{P}_1, \texttt{P}_2, \bar{\texttt{P}}_1, \bar{\texttt{P}}_2);$
else
  return $"Failure"$;
endif;

Theorem. Assume the algorithm is called with a collection of cubes $\mathcal{N}$ and a combinatorial enclosure of f on input. If it does not fail, then it returns representations of an index quadruple of f.

References

- **MM**, Topological invariants, multivalued maps and computer assisted proofs in Dynamics, *Computers Math. Applic.* (**1976**).
- **A. Szymczak**, A combinatorial procedure for finding isolating neighborhoods and index pairs, *Proc. Royal Soc. Edinburgh, Ser. A* (**1997**).

  MM; Index Pairs Algorithms; Foundations of Computational Mathematics; 6;2006; 457-493]