Computational homology in dynamical systems

Marian Mrozek

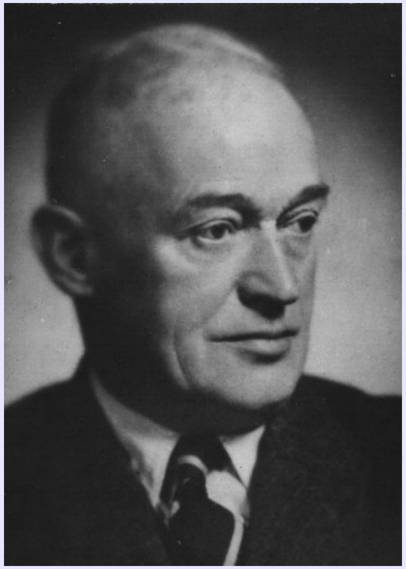
Jagiellonian University, Kraków

National University of Ireland, Galway
29 June - 10 July 2009

Outline

- Dynamical systems
- Rigorous numerics of dynamical systems
- **Homological invariants of dynamical systems**
- Computing homological invariants
- Homology algorithms for subsets of $\mathbb{R}^d$
- Homology algorithms for maps of subsets of $\mathbb{R}^d$
- Applications

Ważewski Theorem



Tadeusz Ważewski,
1896-1972

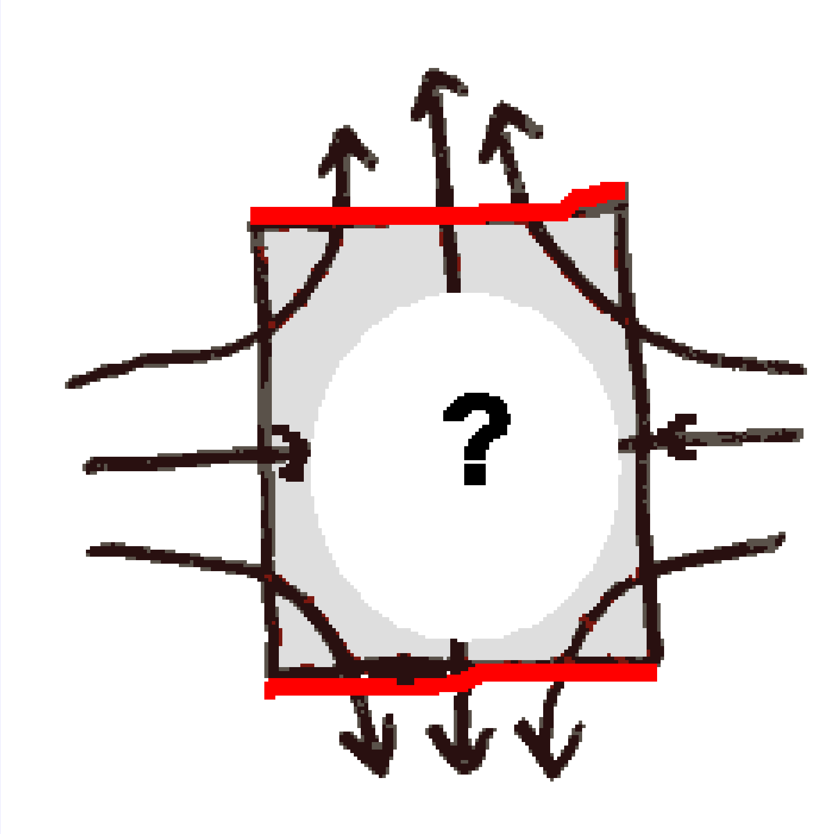
- $A \subset X$ is a deformation retract of X if there exists a map $r : X \to A$ homotopic to identity on X such that $r_{|A} = \mathrm{id}_{|A}$
- The exit set of $N \subset X$, denoted N^-, is

$$\{x \in N \mid \exists \epsilon > 0 \ : \varphi(x,t) \notin N \text{ for } 0 < t < \epsilon\}$$

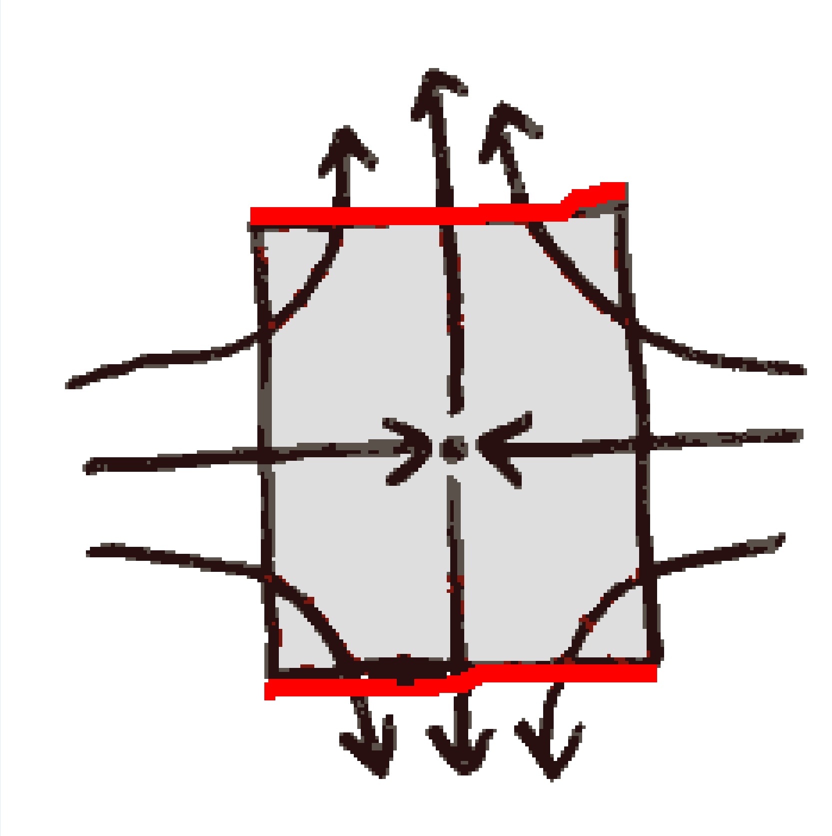
- A compact set N is an isolating block iff N^- is closed

Theorem. (Ważewski, 1947) If N is an isolating block and N^- is not a deformation retract of N, then there exists an $x \in N$ such that $\varphi(x) \subset N$.

Ważewski Theorem



Ważewski Theorem



Isolated invariant sets

- a compact set N is an isolating neighborhood iff

$$x \in \operatorname{bd} N \;\Rightarrow\; \varphi(x) \not\subset N.$$

N is an isolating neighborhood iff $\operatorname{Inv}(N, \varphi) \subset \operatorname{int} N$.

A compact set $S \subset X$ is called an isolated invariant set if there exists an isolating neighborhood N such that $S = \operatorname{Inv}(N, \varphi)$.

Theorem. (Conley and students, 1978)

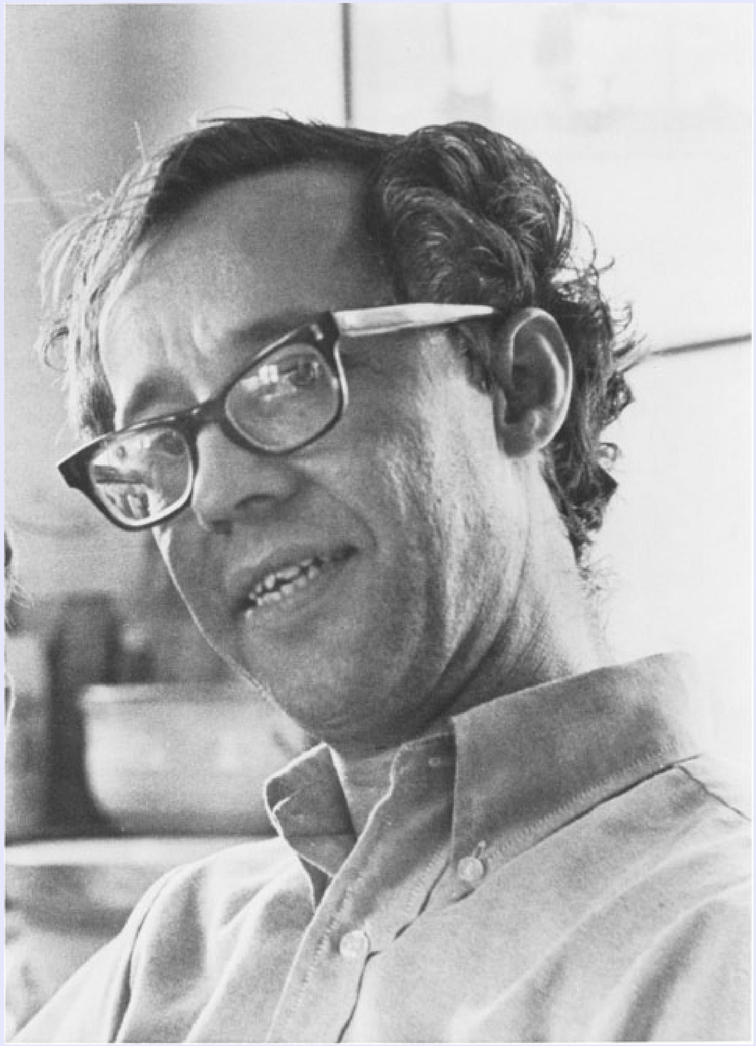
- For every isolating neighborhood N of S there exists an isolating block M such that $S \subset M \subset N$.
- If M_1 and M_2 are two such blocks, then $(M_1/M_1^-, [M_1^-])$ and $(M_2/M_2^-, [M_2^-])$ are homotopy equivalent and, in particular,

$$H^*(M_1, M_1^-) \cong H^*(M_2, M_2^-).$$

The cohomological Conley index of S and N is

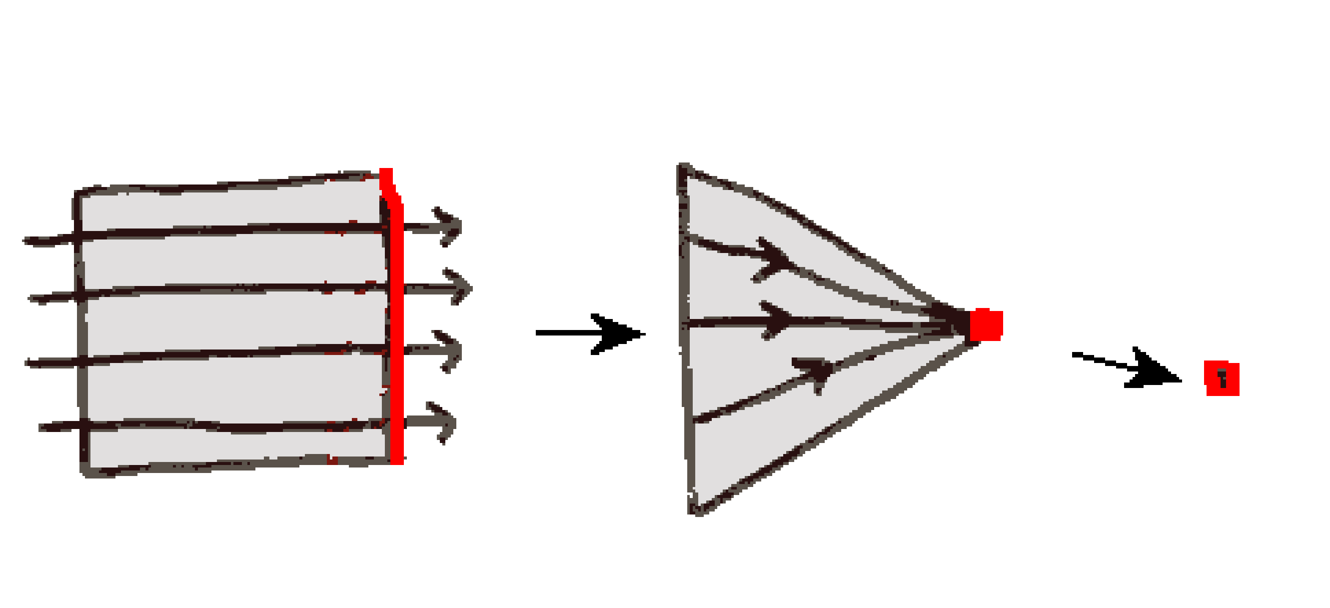
$$\mathrm{Con}^*(N, \varphi) := \mathrm{Con}^*(S, \varphi) := H^*(M, M^-).$$

Charles Conley

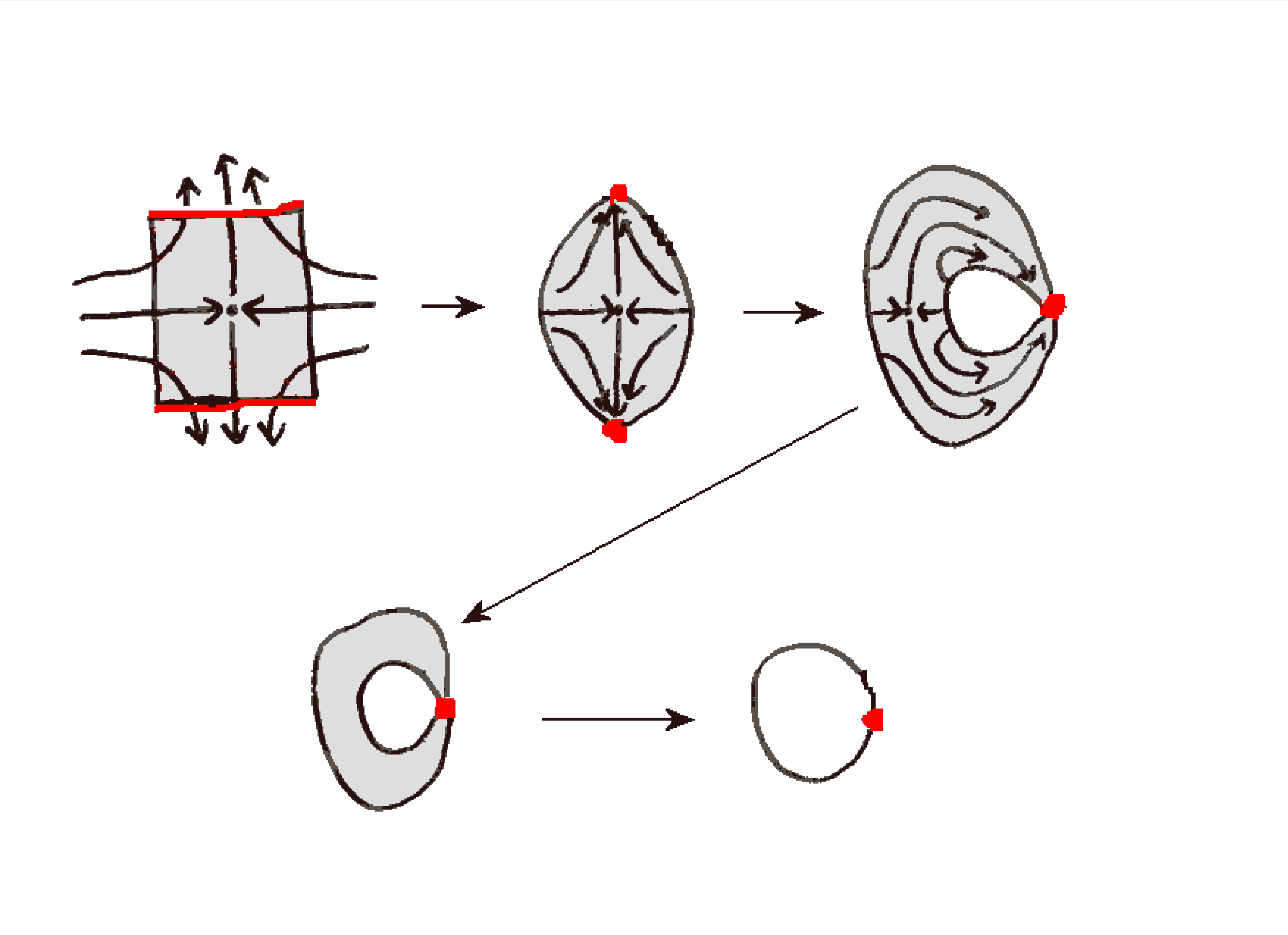


Charles Conley
1933-1984

An example



An example



Main properties

Theorem. (Conley and students, 1978)

- **Ważewski property:**
$$\mathrm{Con}^*(N,\varphi) \neq 0 \;\Rightarrow\; \mathrm{Inv}(N,\varphi) \neq \emptyset.$$
- **Hopf property:** If $\chi(\mathrm{Con}^*(N,\varphi)) \neq 0$ then there exists an $x \in N$ such that $\varphi(x) = \{x\}$.
- **Additivity:** If $S = S_1 \cup S_2$ and $S_1 \cap S_2 \neq \emptyset$ then
$$\mathrm{Con}^*(S,\varphi) = \mathrm{Con}^*(S_1,\varphi) \oplus \mathrm{Con}^*(S_2,\varphi).$$
- **Homotopy invariance:** If N is an isolating neighborhood for a family of flows φ_t continuously depending on t then
$$\mathrm{Con}^*(N,\varphi_0) = \mathrm{Con}^*(N,\varphi_1).$$

Discrete case 12

Let $f : X \to X$ be the generator of φ, i.e. $f(x) := \varphi(x, 1)$.

- A function $\sigma : \mathbb{Z} \to N$ is a solution to f through x in $N \subset X$ if $\sigma(0) = x$ and $f(\sigma(n)) = \sigma(n + 1)$ for all $n \in \mathbb{Z}$.
- invariant part of $N \subset X$:

$$\operatorname{Inv}(N, f) := \{\, x \in N \mid \exists \sigma : \mathbb{Z} \to N \text{a solution to } f \text{ through } x \,\}$$

- A compact set $S \subset X$ is called an isolated invariant set for f if there exists a compact neighborhood N of S such that $S = \operatorname{Inv}(N, \varphi) \subset \operatorname{int} N$.
- Then, N is called an isolating neighborhood (for S).

Index pairs

A pair of compact sets $P = (P_1, P_2)$ is called an index pair for f and an isolated invariant set S iff

(i) (positive relative invariance)

$$f(P_2) \cap P_1 \subset P_2$$

(ii) (exit set)

$$P_1 \cap \mathrm{cl}(f(P_1) \setminus P_1) \subset P_2$$

(iii) (isolation)

$$S = \mathrm{Inv}(\mathrm{cl}(P_1 \setminus P_2), f) \subset \mathrm{int}(P_1 \setminus P_2)$$

Index pairs

Proposition. If pair $P = (P_1, P_2)$ of compact subsets of an isolating neighborhood N satisfies

$$f(P_2) \cap P_1 \subset P_2$$

$$P_1 \setminus f^{-1}(N) \subset P_2$$

$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \setminus P_2).$$

then P is an index pair for f and $\operatorname{Inv}(N, f)$.

$H^*(P_1, P_2)$ is not an invariant.

Normal functors

- $\mathcal{E}$ - a category
- the category of endomorphisms of $\mathcal{E}$:
  - Objects: pairs (E, e), where $A \in \mathcal{E}$ and $e \in \mathcal{E}(E, E)$
  - Morphisms: $\psi\,(E_1, , e_1) \to (E_2, e_2)$ iff $\psi \in \mathcal{E}(E_1, E_2)$ and $\psi e_1 = e_2 \psi$
- the category of automorphisms of $\mathcal{E}$ - full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of pairs $(A, a) \in \mathrm{Endo}(\mathcal{E})$ such that $a \in \mathcal{E}(A, A)$ is an automorphism
- $L : \mathrm{Endo}(\mathcal{E}) \to \mathcal{C}$ is a normal functor if $L(e)$ is an isomorphism in $\mathcal{C}$ for any endomorphism $e \in \mathcal{E}(E, E)$.

Leray Functor

- $\mathcal{V}_0$ - the category of finitely dimensional vector spaces over $\mathbb{R}$
- the generalized kernel of $\alpha \in \mathcal{V}_0(V, V)$ is
$$\operatorname{gker} \alpha := \bigcup_{n \in \mathbb{N}} \ker \alpha^n$$
- the Leray functor $L : \mathrm{Endo}(\mathcal{V}_0) \longrightarrow \mathrm{Auto}(\mathcal{V}_0)$:
$$L(V, v) := (V / \operatorname{gker} v, v')$$

Leray functor is normal .

Index quadruples and index maps

A quadruple $P = (P_1, P_2, \bar{P}_1, \bar{P}_2)$ is an index quadruple for f and S if (P_1, P_2) is an index pair for f and S and $(\bar{P}_1, \bar{P}_2)$ is a topological pair such that the map

$$f_P : (P_1, P_2) \ni x \to f(x) \in (\bar{P}_1, \bar{P}_2)$$

$$\iota_P : (P_1, P_2) \ni x \to x \in (\bar{P}_1, \bar{P}_2)$$

are well defined and ι_P is an excision (induces an isomorphism in cohomology)

Given an index quadruple, we define the index map as the composition

$$I_P := H^*(f_{P\bar{P}}) \circ H^*(\iota_P)^{-1}$$

The Conley index for discrete dynamical systems

Theorem. (MM,1990,2005) For every isolating neighborhood N of f there exists an index quadruple P such that

$$\mathrm{Inv}(N, f) \subset P_1 \subset \bar{P}_1 \subset N.$$

Moreover, if P and Q are two such quadruples, then

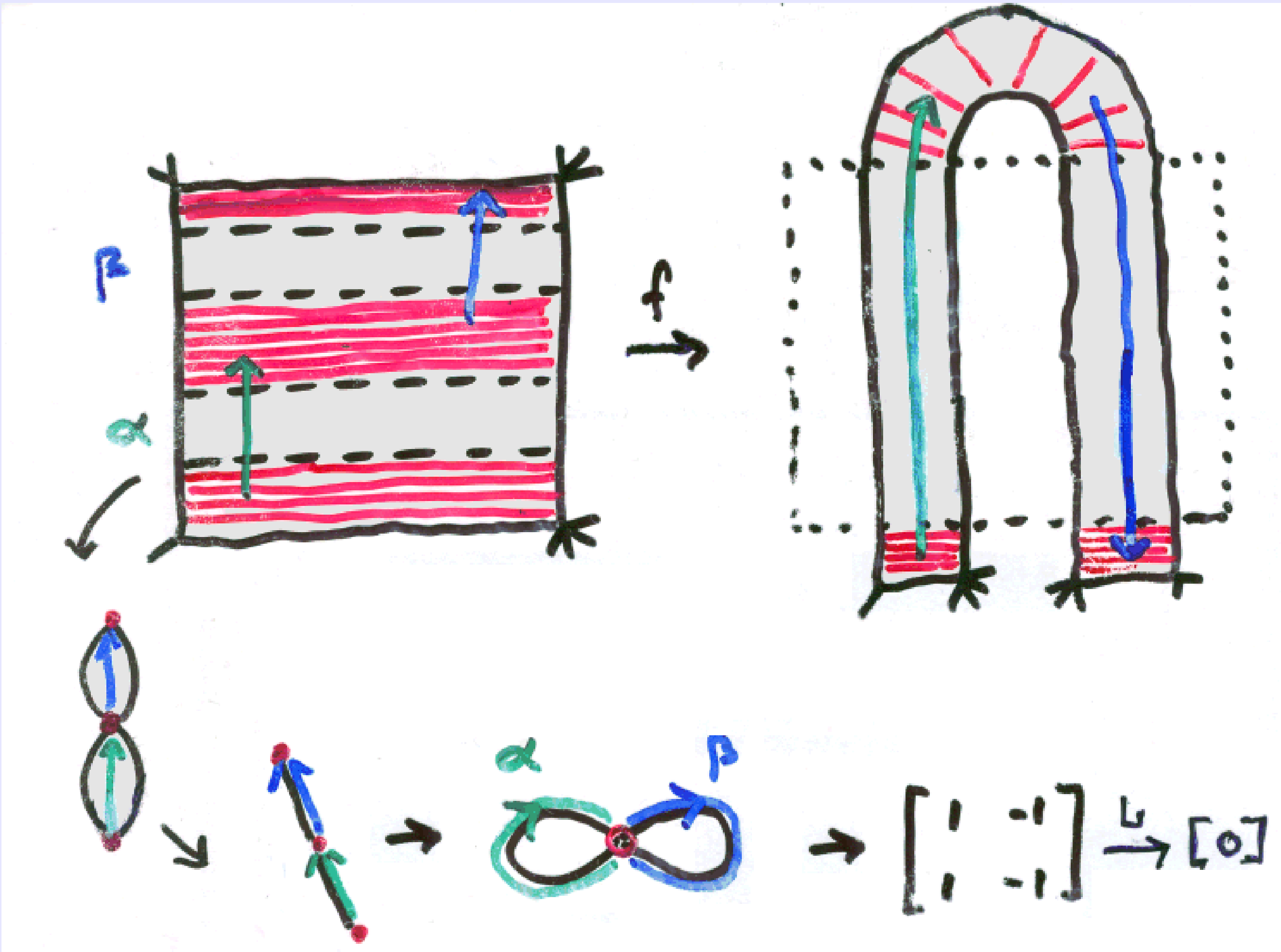
$$L(H^*(P_1, P_2), I_P) \cong L(H^*(Q_1, Q_2), I_Q).$$

The Conley index of f in N is

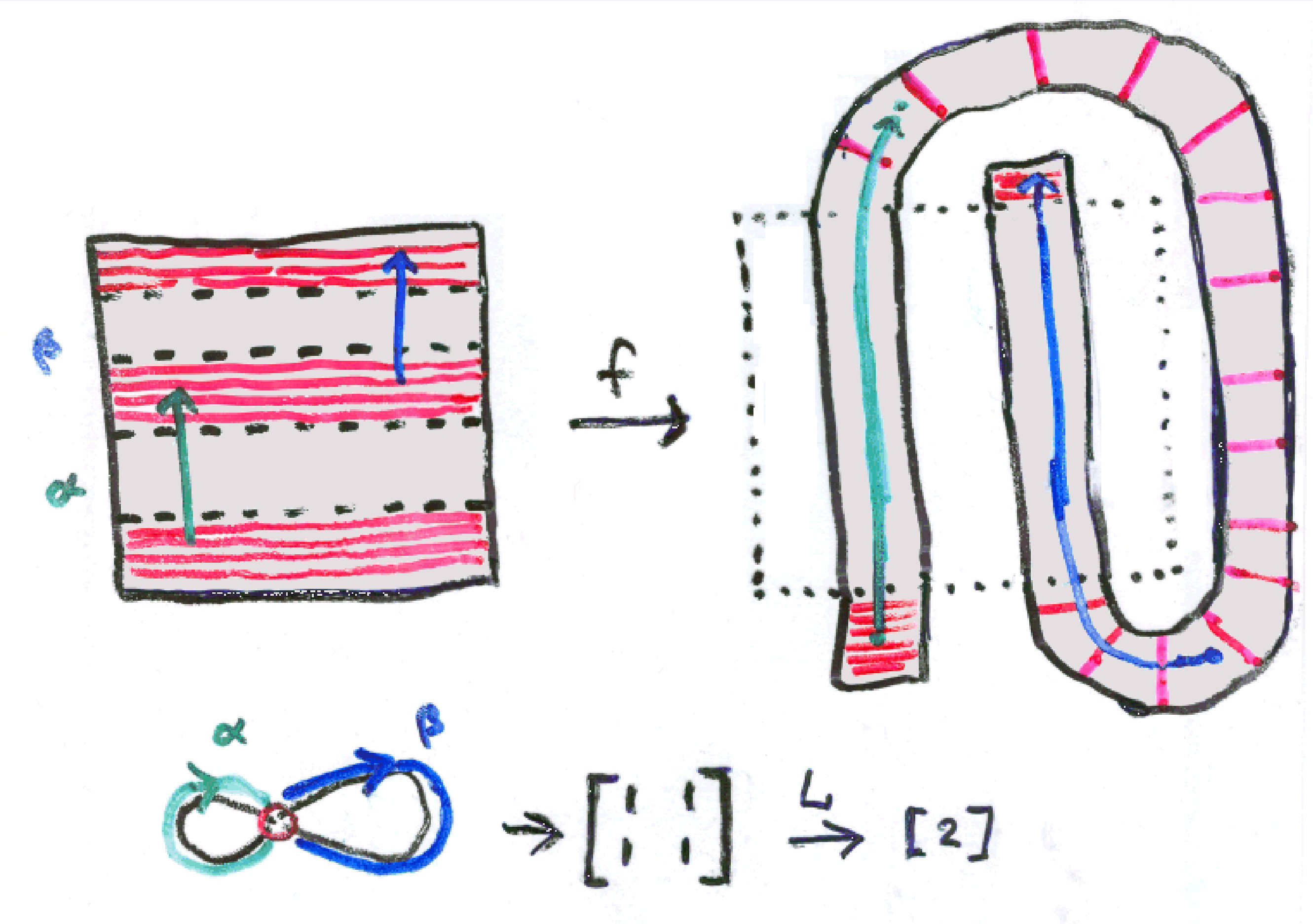
$$(CH^*(N, f), \chi(N, f)) := L(H^*(P_1, P_2), I_P).$$

- J.W. Robbin, D. Salamon, 1988 - shape theory, inverse limit functor
- MM, 1990 - cohomology, Leray functor
- A. Szymczak, 1995 - homotopy, Szymczak functor (most general)
- J. Franks, D. Richeson, 2000 - a reformulation of Szymczak construction in terms of shift equivalence

An example [20]



An example



Main properties

Theorem.

- **Ważewski property** (J.W. Robbin, D. Salamon, 1988):
$$\mathrm{Con}^*(N,f) \neq 0 \;\Rightarrow\; \mathrm{Inv}(N,f) \neq \emptyset.$$
- **Lefschetz property** (MM, 1989): If $\Lambda(\chi(N,f)) \neq 0$ then there exists an $x \in N$ such that $f(x) = x$.
- **Additivity:**(J.W. Robbin, D. Salamon, 1988): If $S = S_1 \cup S_2$ and $S_1 \cap S_2 \neq \emptyset$ then
$$\mathrm{Con}^*(S,f) = \mathrm{Con}^*(S_1,f) \oplus \mathrm{Con}^*(S_2,f).$$
- **Homotopy invariance:**(J.W. Robbin, D. Salamon, 1988): If N is an isolating neighborhood for a family of flows f_t continuously depending on t then
$$\mathrm{Con}^*(N,f_0) = \mathrm{Con}^*(N,f_1).$$

Discrete vs. continuous case.

Theorem. (MM, 1990) Let $\varphi : X \times \mathbb{R} \to X$ be a flow and for $t \in \mathbb{R}$ let $\varphi_t : X \to X$ be the map defined by

$$\varphi_t(x) := \varphi(x, t).$$

If $S \subset X$ is a compact set, then the following conditions are equivalent.

(i) S is an isolated invariant set with respect to φ,

(ii) S is an isolated invariant set with respect to φ_t for all $t \neq 0$,

(iii) S is an isolated invariant set with respect to φ_t for some $t \neq 0$.

Moreover, if one of the above conditions is satisfied, then for any $t \neq 0$

$$\chi(N, \varphi_t) = \mathrm{id},$$
$$\mathrm{Con}^*(N, \varphi_t) \cong \mathrm{Con}^*(N, \varphi).$$

Conley index and horseshoe dynamics

Given a compact set N and $\alpha \in \{0,1\}^n$ put

$$N_\alpha := \bigcap_{i=0}^{n-1} f^i(N_{\alpha_i})$$

and for $\bar{\alpha} = (\alpha^1, \alpha^2, \ldots \alpha^m)$ with $\alpha^j \in \{0,1\}^n$ put

$$N_{\bar{\alpha}} := \bigcup_{j=1}^{m} N_{\alpha^j}.$$

Proposition. If N is an isolating neighborhood for f then so is N_α and $N_{\bar{\alpha}}$

Conley index and horseshoe dynamics

Theorem. (K. Mischaikow, MM, 1993) Assume

$$N = N_0 \cup N_1$$

is an isolating neighbourhood for f such that N_0 and N_1 are disjoint compact polyhedra. If for $k = 0, 1$

$$\mathrm{Con}^n(N_k) = \begin{cases} (\mathbf{Q}, \mathrm{Id}) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $\chi^*(N_{00,01,11}, f)$, $\chi^*(N_{00,10,11}, f)$ are different from identity then there exists a $d \in \mathbf{N}$ and a continuous surjection

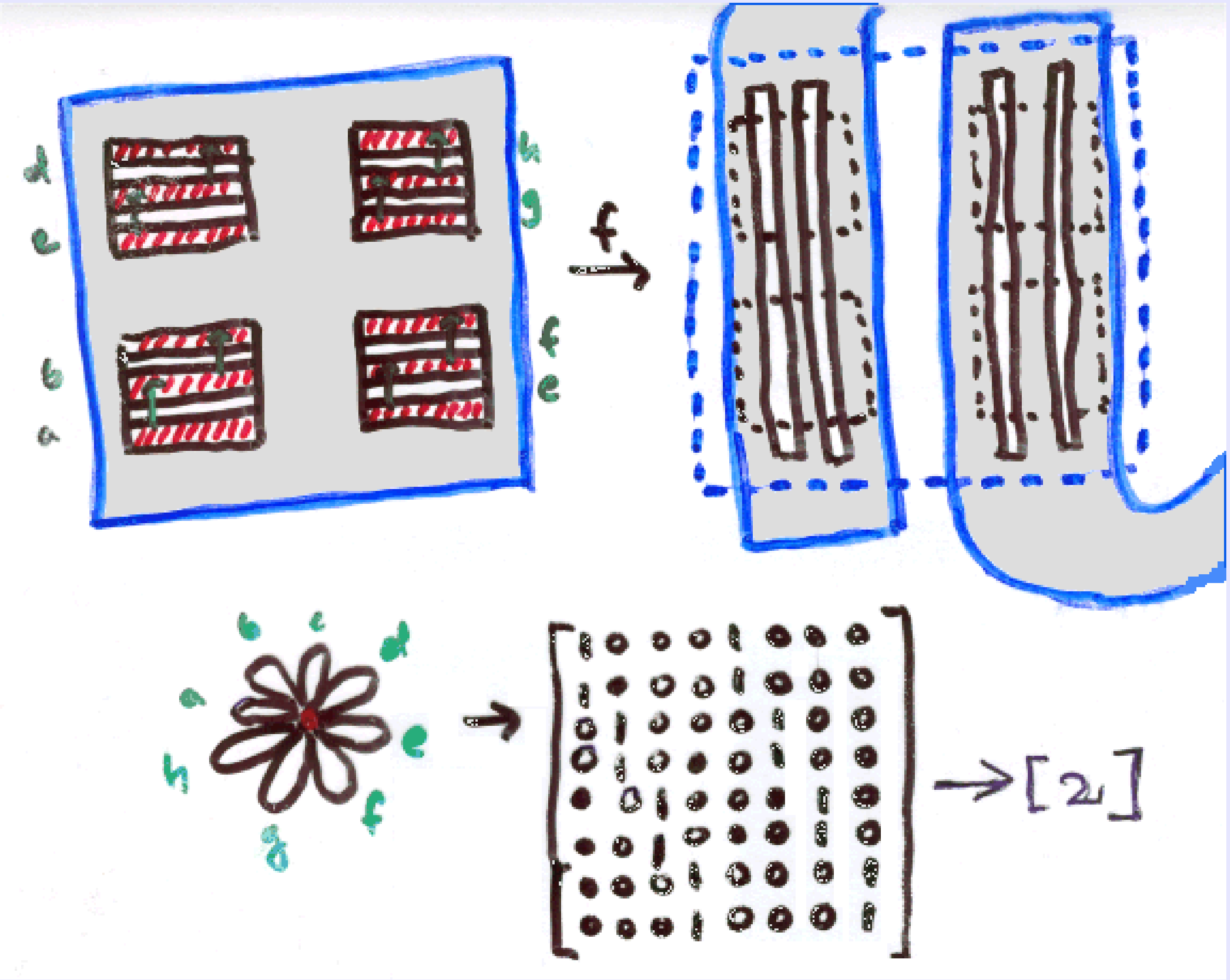
$$\rho : \mathrm{Inv}(N, f) \to \Sigma_2$$

such that

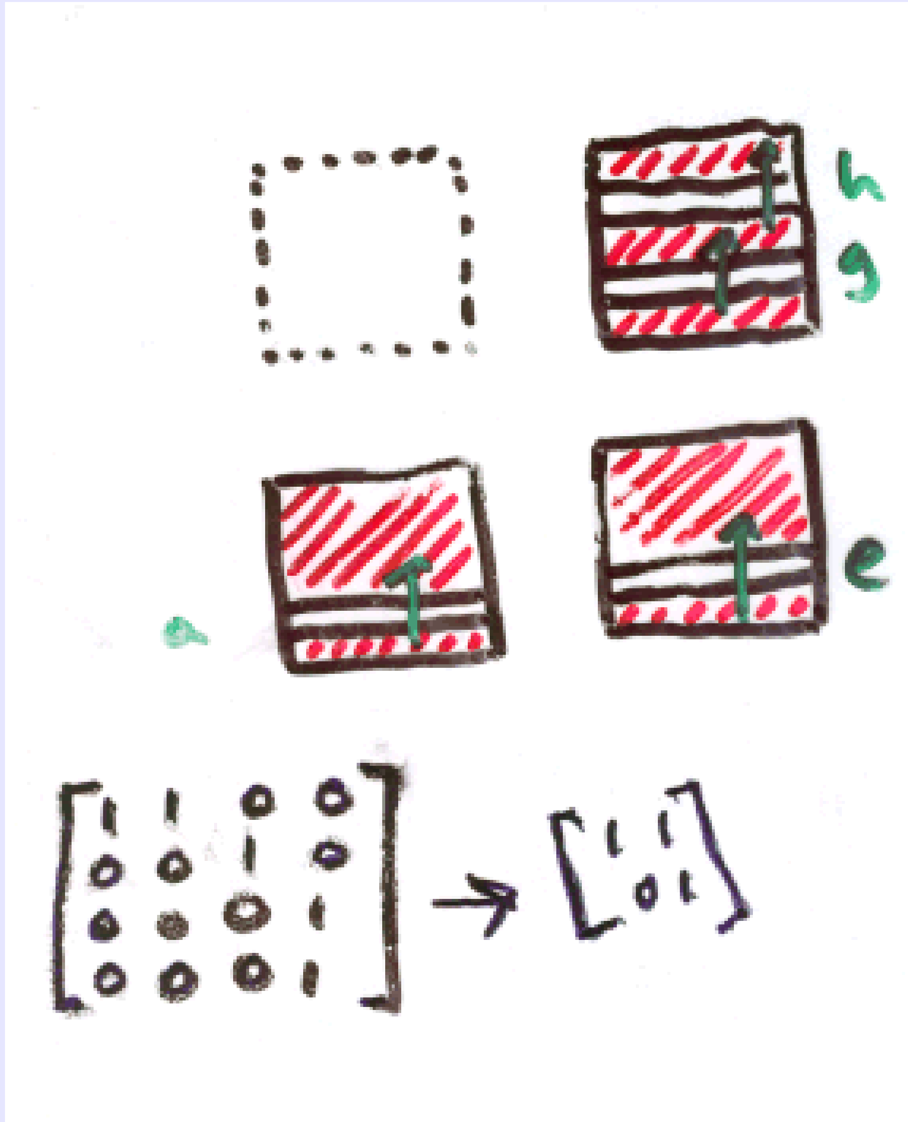
$$\rho \circ f^d = \sigma \circ \rho$$

where $\sigma : \Sigma_2 \to \Sigma_2$ is the full shift dynamics on two symbols. Moreover, for each periodic sequence $\alpha \in \Sigma_2$ there exists a periodic point $x \in N$ such that $\rho(x) = \alpha$.

An example



An example



An example

References

- **T. Ważewski**, Une méthode topologique de l'examen du phénomène asymptotique relativement aux équations différentielles ordinaires, *Rend. Accad. Nazionale dei Lincei, Cl. Sci. fisiche, mat. e naturali* (**1947**).
- **C. Conley**, Isolated invariant sets and the Morse index, *CBMS Regional Conference Series in Math* (**1978**).
- **J. Robbin and D. Salamon**, Dynamical systems, shape theory, and the Conley index, *Ergodic Theory and Dynamical Systems* (**1988**).
- **MM**, Leray functor and the cohomological Conley index for discrete time dynamical systems, *Trans. Amer. Math. Soc.* (**1990**).
- **A. Szymczak**, The Conley index for discrete semidynamical systems, *Topology and Its Applications* (**1995**).
- **J. Franks and D. Richeson**, Shift equivalence and the Conley index, *Trans. Amer. Math. Soc.* (**2000**).