Computational homology in dynamical systems

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Outline

- Dynamical systems
- Rigorous numerics of dynamical systems
- Homological invariants of dynamical systems
- Computing homological invariants
- Homology algorithms for subsets of $\mathbb{R}^d$
- Homology algorithms for maps of subsets of $\mathbb{R}^d$
- Applications

Ghost solutions 3

Consider the equation

$$z' = (\alpha i - |z|)z, \quad z \in \mathbb{C}$$

The only periodic trajectory of this equation is the stationary point at the origin.
Consider its Euler discretization

$$\Phi_h(z) := z(1 + h(\alpha i - |z|))$$

For every $h > 0$ this discretization has invariant circles of radius

$$r_\pm := \frac{1 \pm \sqrt{1 - h^2\alpha^2}}{h}$$

Disappearing Smale's horseshoe

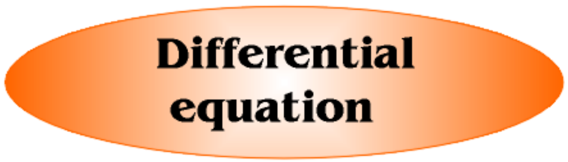
- The logistic equation
$$y' = y(1-y)$$
may be solved explicitly and it clearly does not exhibit chaotic behaviour
- However, Koçak and Hale (1991) prove that the two step numerical scheme
$$\Phi_{h,\lambda}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} \frac{1-\lambda}{1+\lambda} y_2 + \frac{2\lambda}{1+\lambda} y_1 + 2h y_1(1-y_2) \\ y_1 \end{pmatrix}$$
contains an invariant subset conjugate to a horseshoe for every $h > 0$.

Fatal consequences of numerical errors

- On 25th of February 1991 the Patriot missile system failed to intercept an Iraqi Scud and 28 American soldiers were killed.

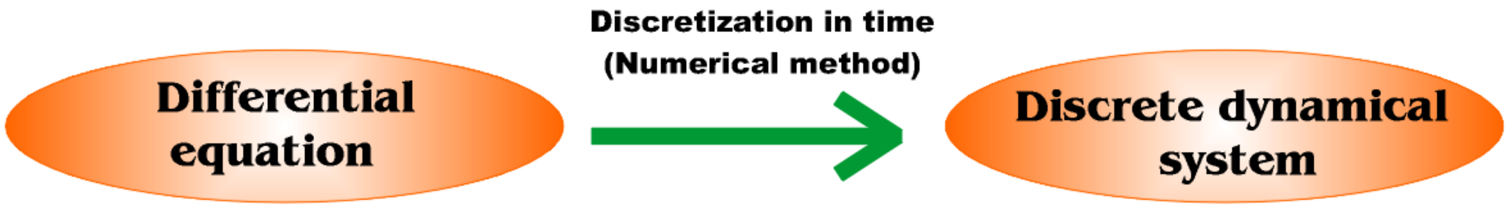
- Ariane 5's first test flight on 4 June 1996 failed, with the rocket self-destructing 37 seconds after launch because of a malfunction in the control software, resulting in a loss of more than $370 million.

In both cases the failures were attributed to numerical errors.

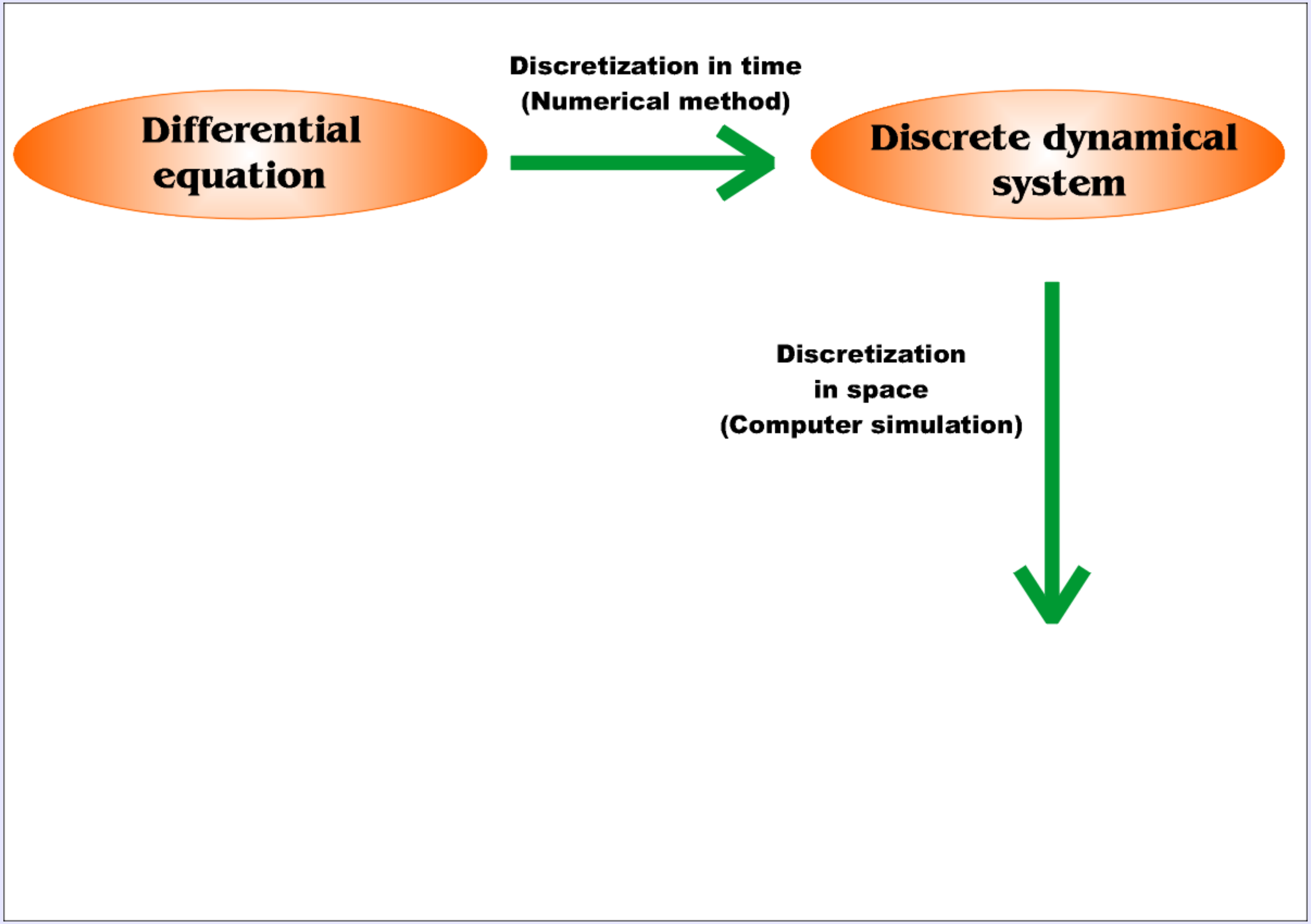

Differential
equation


Differential equation
Discretization in time
(Numerical method)

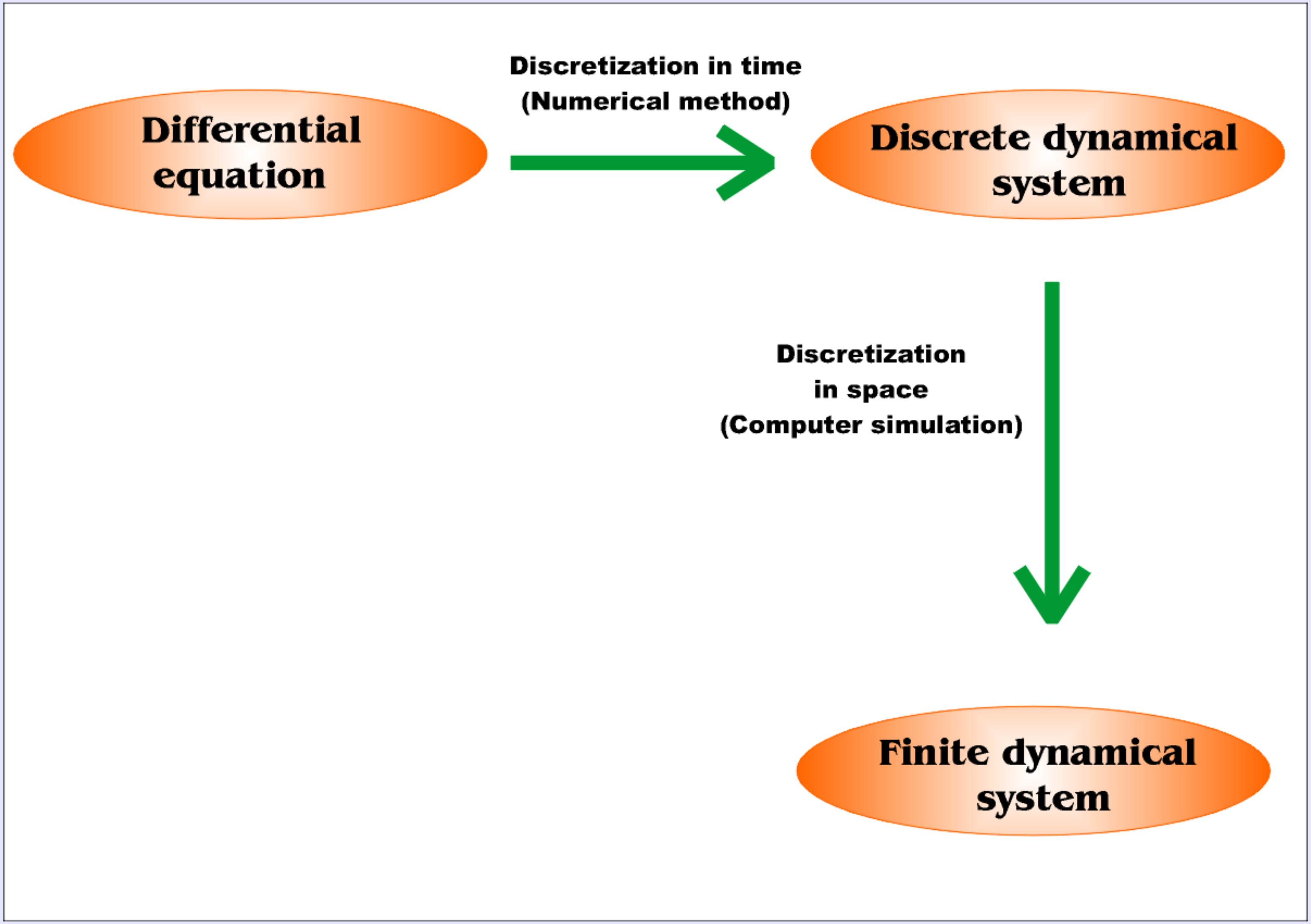
Numerical Analysis of Dynamical Systems



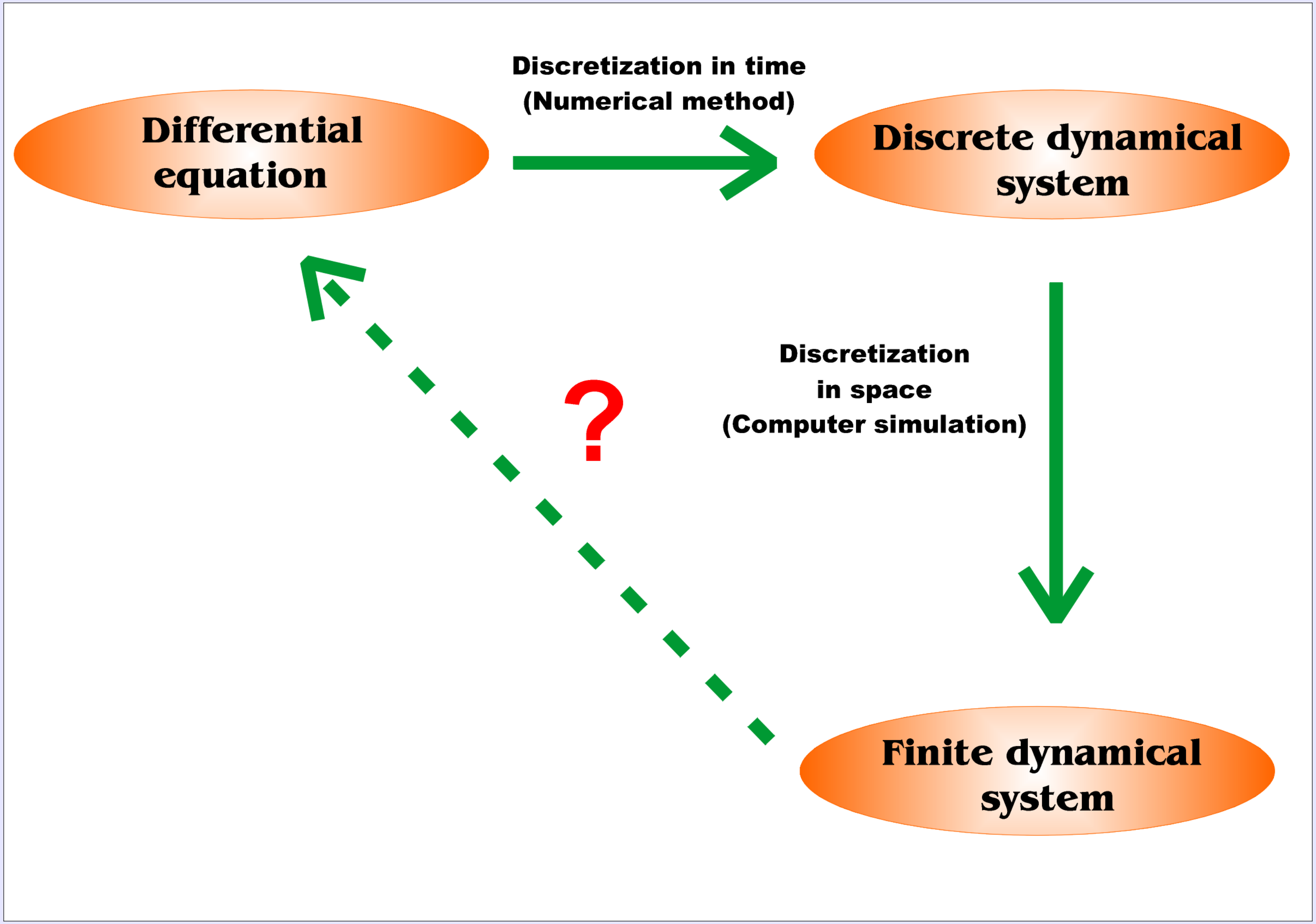
Numerical Analysis of Dynamical Systems



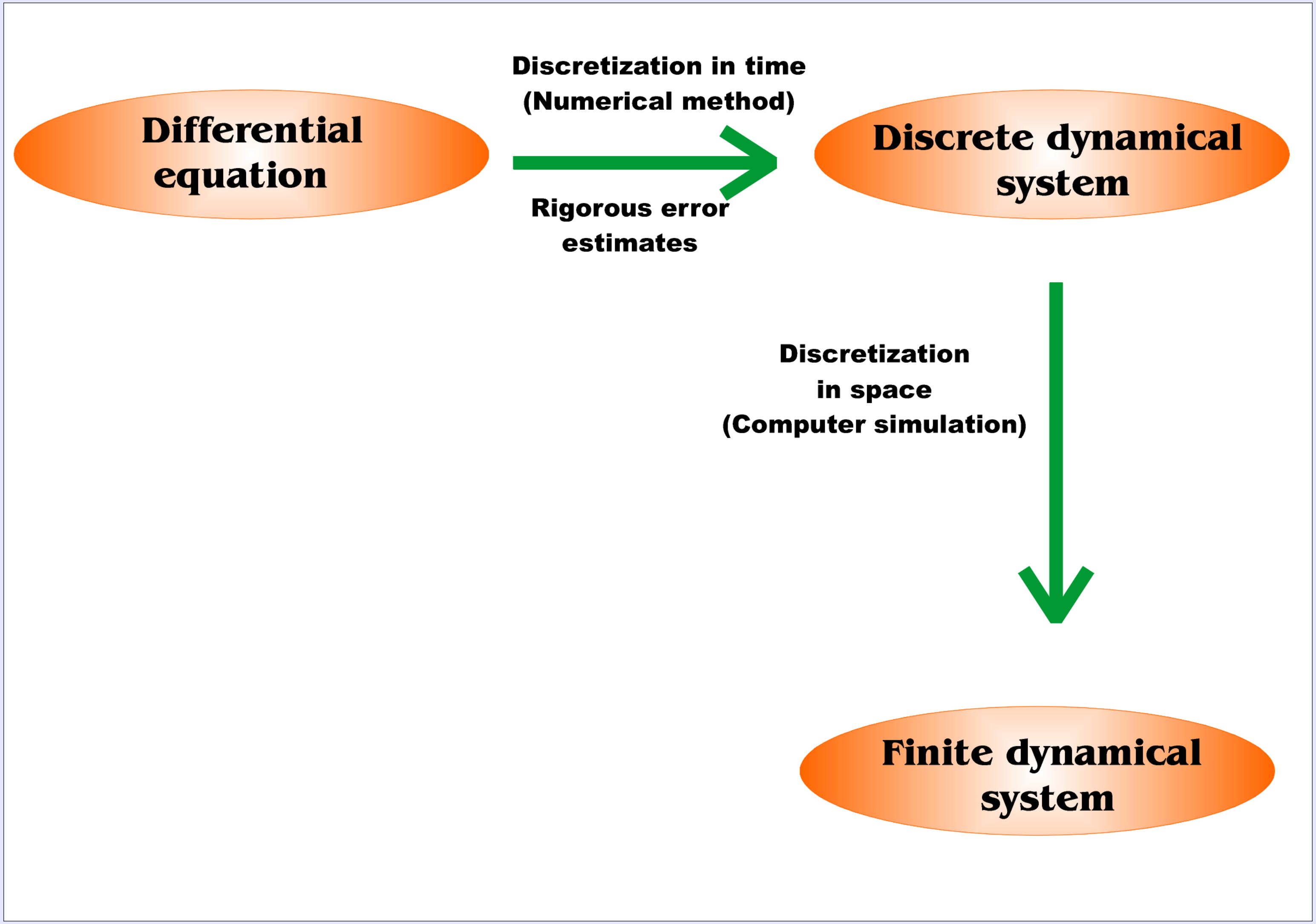
Numerical Analysis of Dynamical Systems



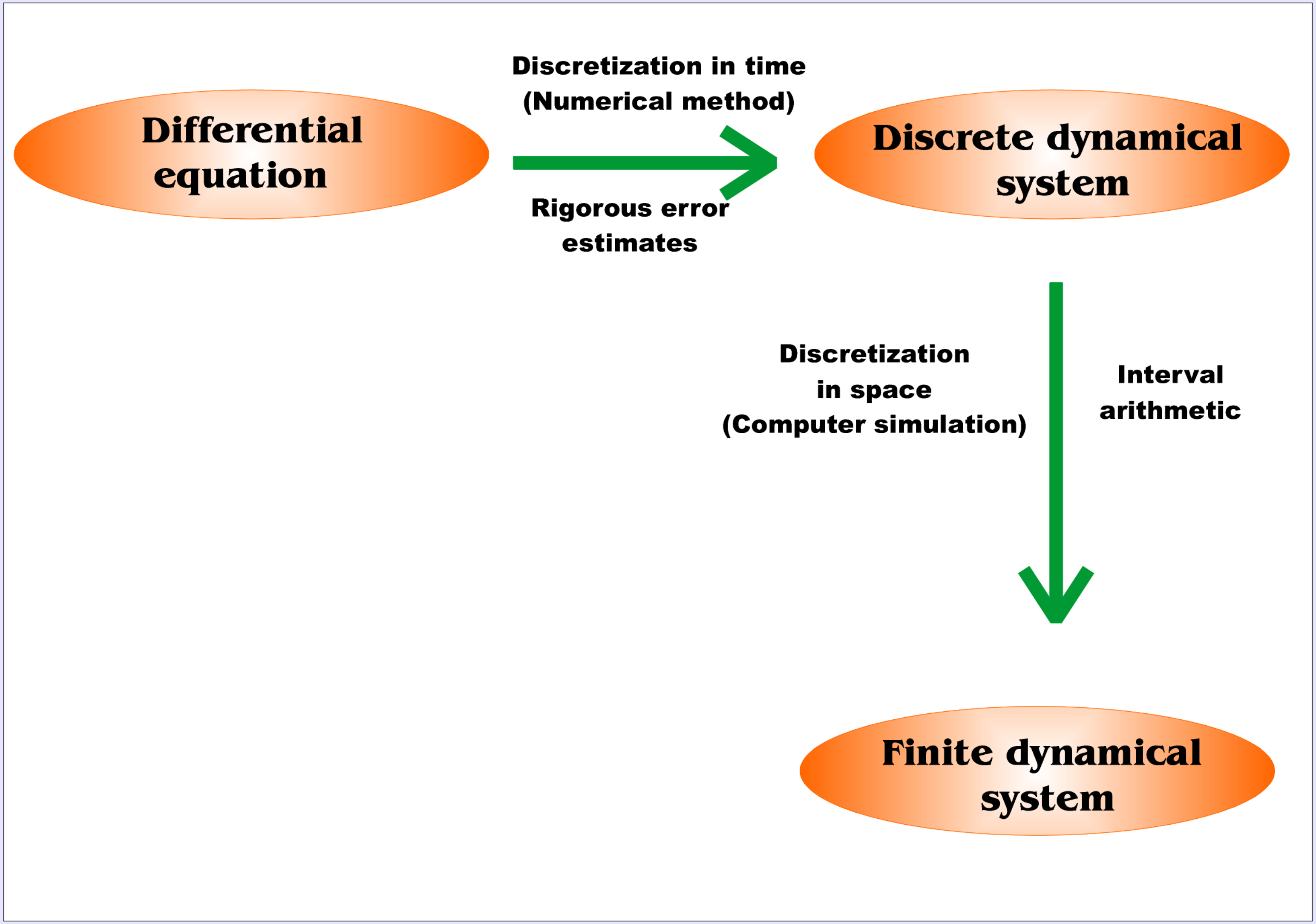
Numerical Analysis of Dynamical Systems



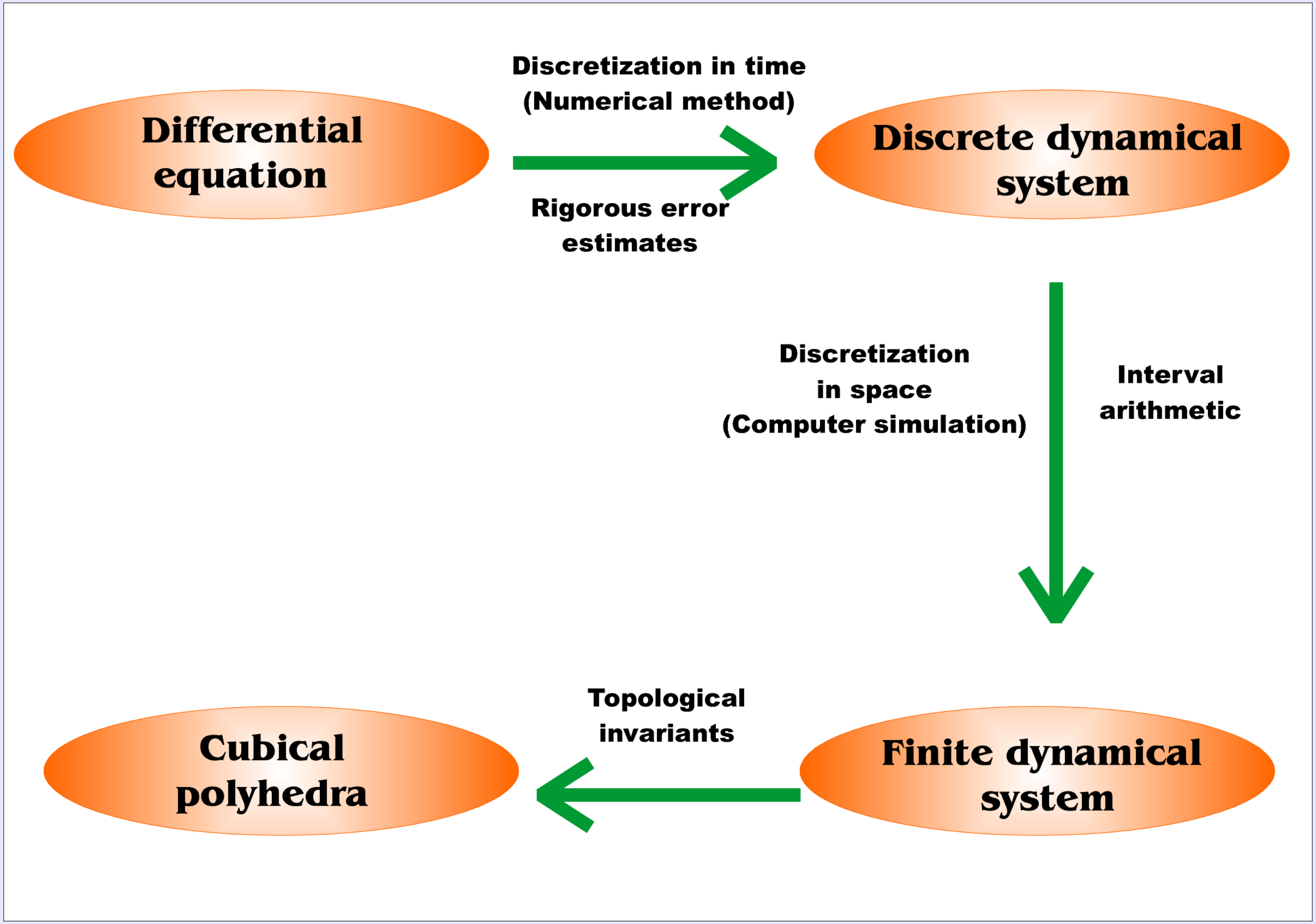
Numerical Analysis of Dynamical Systems

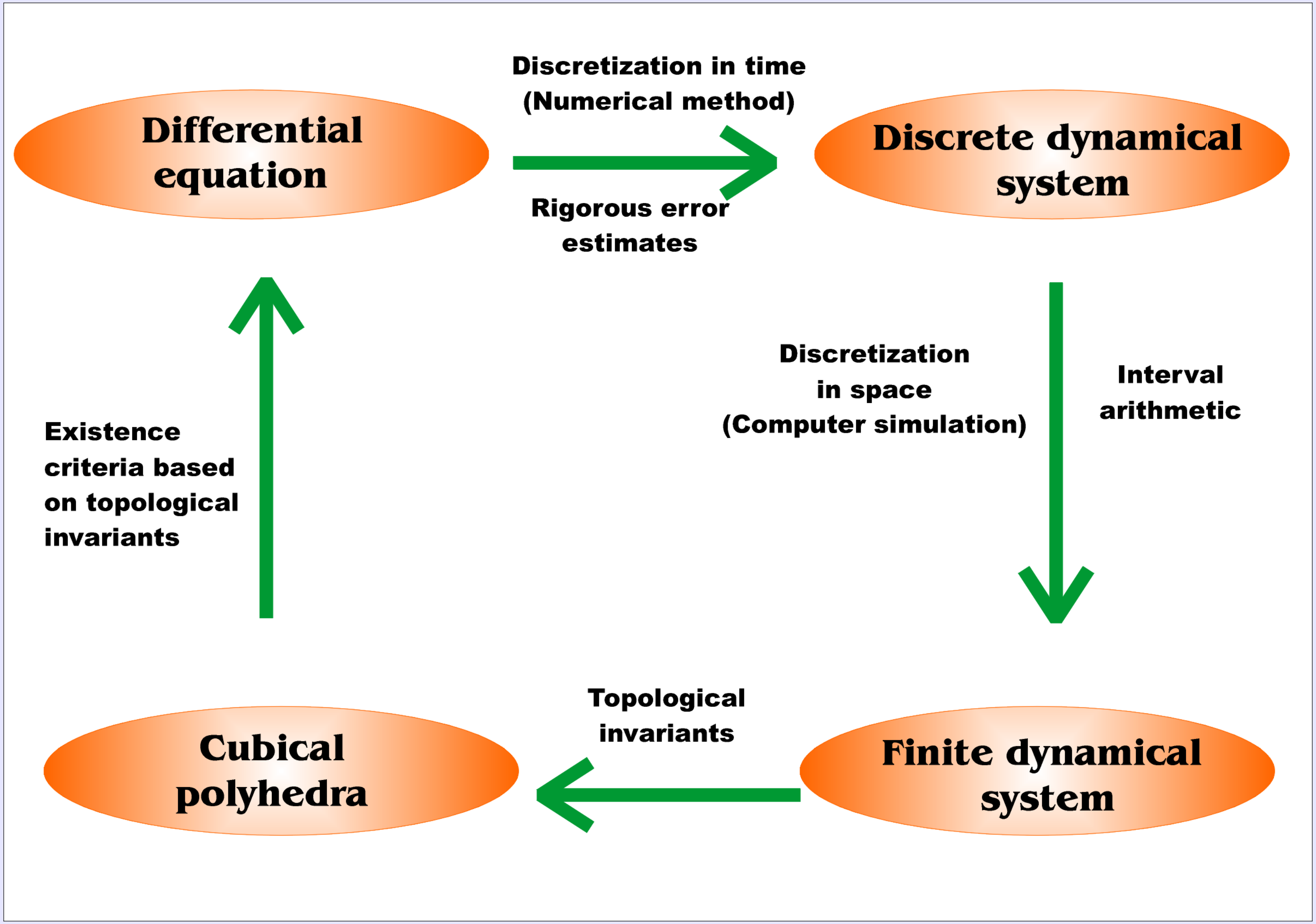


Numerical Analysis of Dynamical Systems



Numerical Analysis of Dynamical Systems




Differential equation
Discretization in time (Numerical method)
Rigorous error estimates
Discrete dynamical system
Discretization in space (Computer simulation)
Interval arithmetic
Finite dynamical system
Topological invariants
Cubical polyhedra
Existence criteria based on topological invariants

Chaos in Lorenz equations

Theorem. (K. Mischaikow, MM, 1995) Consider the Lorenz equations

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = Rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

and put

$$P := \{(x,y,z) \in \mathbb{R}^3 \mid z = 53\}.$$

For all parameter values in a sufficiently small neighborhood of $(\sigma, R, b) = (45, 54, 10)$, there exists a Poincaré section $N \subset P$ such that the Poincaré map g induced by (1) is Lipschitz and well defined. Furthermore, there exists a $d \in \mathbb{N}$ and a continuous surjection $\rho : \mathrm{Inv}(N, g) \longrightarrow \Sigma_2$ such that

$$\rho \circ g^d = \sigma \circ \rho$$

where $\sigma : \Sigma_2 \longrightarrow \Sigma_2$ is the full shift dynamics on two symbols.

Rigorous numerics of dynamical systems

- Goal: use the outcome of numerical simulations to draw rigorous conclusions about the behaviour of the original dynamical system.
- Needed:
  (1) exact bounds for the errors resulting from the time discretization and space discretization
  (2) a method to draw conclusions about the original dynamical system from the outcome of numerical simulations

Interval arithmetic

- $\hat{\mathbb{R}} \subset \bar{\mathbb{R}}$ — fixed, finite set of representable numbers
- representable intervals:

$$\mathcal{I} := \{[a, b] \mid a, b \in \hat{\mathbb{R}}, a \leq b\}$$

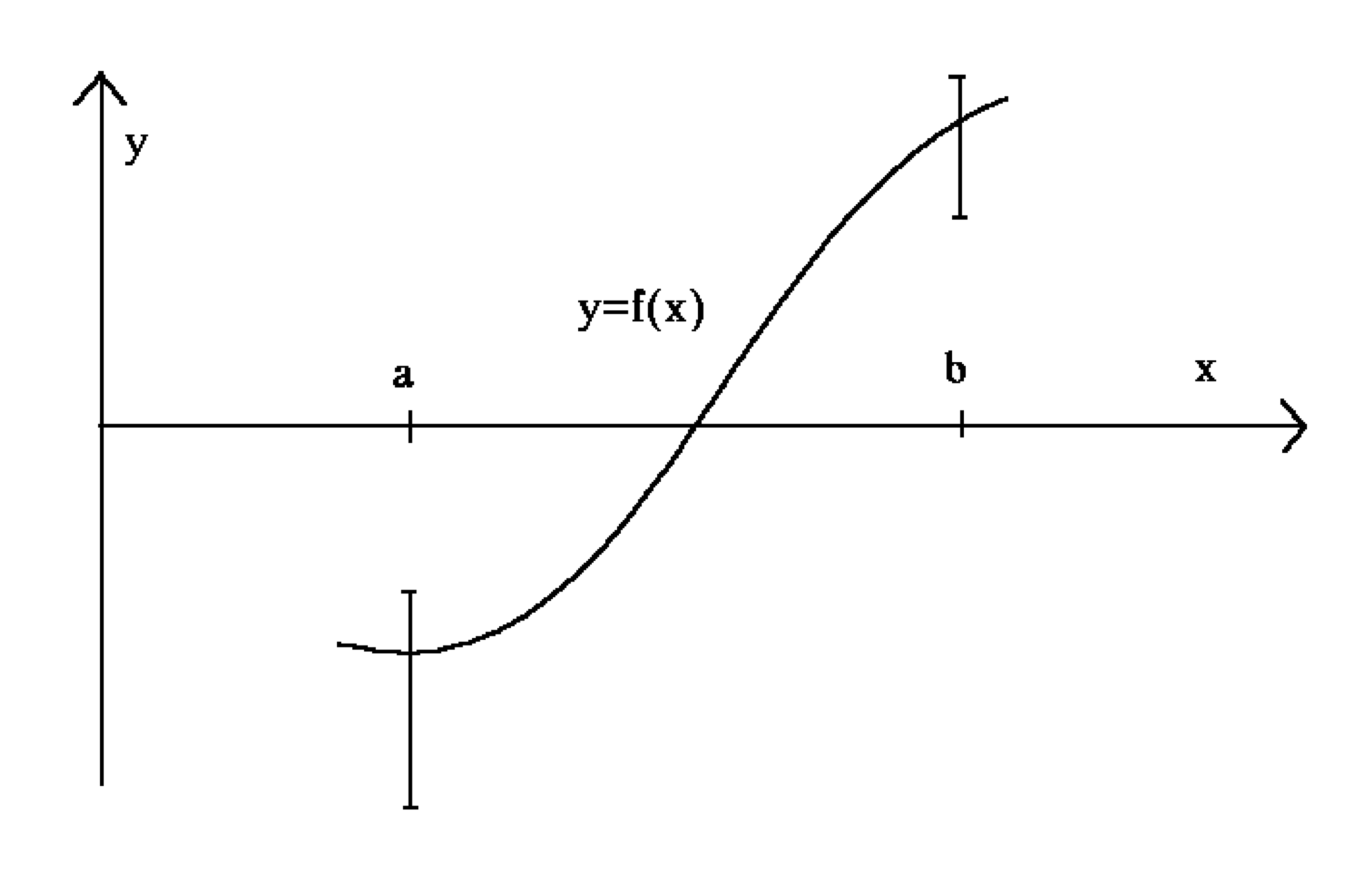
- For $\diamond \in \{+, -, *, /\}$ and $I, J \in \mathcal{I}$ denote by $I \diamond J$ the smallest representable interval that contains

$$\{a \diamond b \mid a \in I, b \in J\}.$$

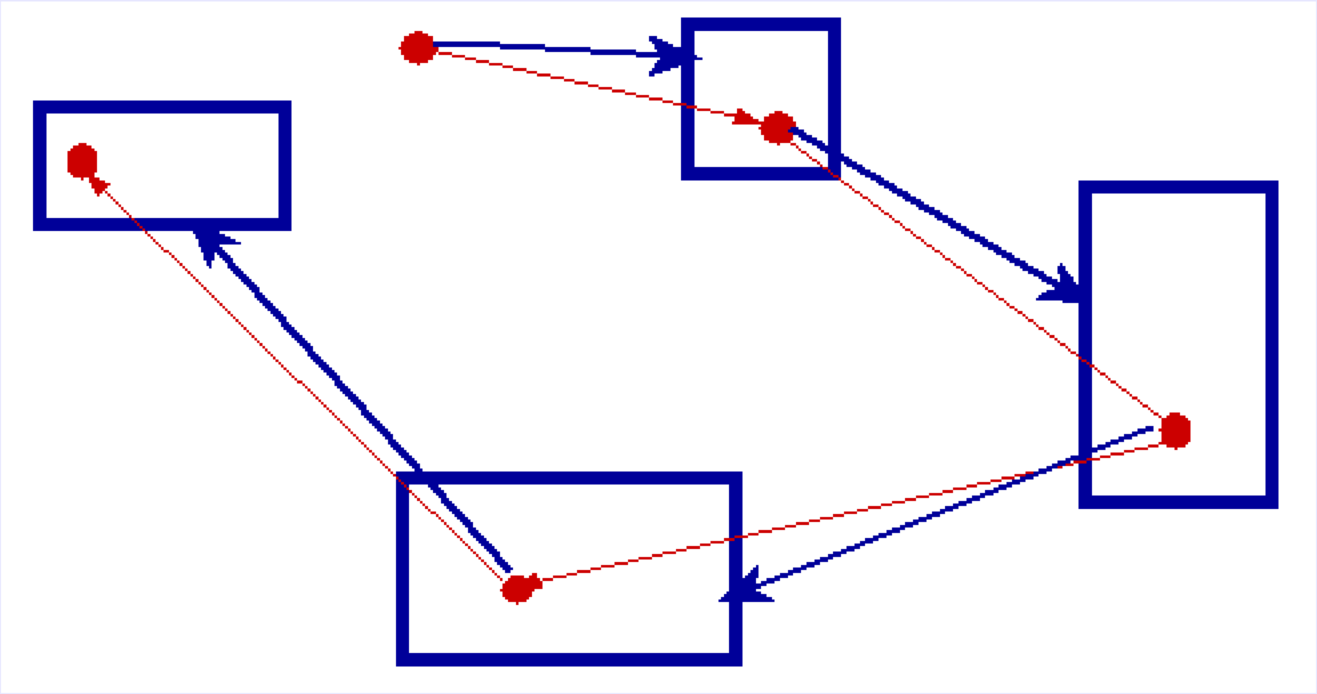
Given the endpoints of I and J, one can easily construct the endpoints of $I \diamond J$.

- first proposed by M. Warmus in 1956
- rediscovered by R.E. Moore in 1959

The simplest topological tool: Darboux property



Advanced tool: topology of multivalued maps



Multivalued maps

Let X, Y be topological spaces. A multivalued map $F : X \rightrightarrows Y$ from X to Y is a function $F : X \to 2^Y$ from X to subsets of Y.

- The image of $A \subset X$ is
$$F(A) := \bigcup_{x \in A} F(x).$$
- The weak preimage of $B \subset Y$ under F is
$$F^{-1}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

F is upper semicontinuous if $F^{-1}(B)$ is closed for any closed set $B \subset Y$, and it is lower semicontinuous if the set $F^{-1}(U)$ is open for any open set $U \subset Y$.

Representations of rational functions

- $f : \mathbb{R}^m \multimap \mathbb{R}^n$ — a rational function.
- $[f] : \mathcal{I}^m \multimap \mathcal{I}^n$, the interval extension of f obtained by replacing the arithmetic operations in f with their interval counterparts

Proposition. Assume $f : \mathbb{R}^m \multimap \mathbb{R}^n$ is a rational function. Then for any $\mathbf{x}_1, \ldots \mathbf{x}_n \in \operatorname{dom}[f]$ we have

$$f(\mathbf{x}_1, \ldots \mathbf{x}_n) \subset [f](\mathbf{x}_1, \ldots \mathbf{x}_n).$$

Arbitrary functions

- $f : \mathbb{R}^m \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{R}^n$
- $g : \mathbb{R}^m \mathrel{-\!\!\!\circ\!\!\!\rightarrow} \mathbb{R}^n$ — a rational approximation of f such that for $x \in D$ and some $\mathbf{w} \in \mathcal{I}^n$

$$f(x) - g(x) \in \mathbf{w},$$

- then

$$f(\mathbf{x}) \subset [g](\mathbf{x})[+]\mathbf{w}.$$

References

- **M. Warmus**, Calculus of approximations, *Bull. de l'Académie Polonaise des Sciences Cl. III)* (**1956**).
- **R.E. Moore**, Interval analysis, *Prentice-Hall* (**1966**).
- **J.K. Hale and H. Koçak**, Dynamics and Bifurcations, *Springer-Verlag* (**1991**).
- **K. Mischaikow, MM**, Chaos in Lorenz equations: a computer assisted proof, *Bull. Amer. Math. Soc. (N.S.)* (**1995**).
- **K. Mischaikow, MM**, Chaos in the Lorenz equations: a computer assisted proof. Part II: details, *Mathematics of Computation* (**1998**).
- **A.M. Stuart, A.R. Humphries**, Dynamical Systems and Numerical Analysis, *Cambridge Univ. Press* (**1998**).