Computational homology in dynamical systems

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Weather forecasts for Galway, Ireland from Weather Underground [2]

Extended Forecast

Updated: 1:00 AM IST on June 20, 2009

Wednesday Night
Chance of Rain. Scattered Clouds. Low: 12 °C . Wind ESE 14 km/h . Chance of precipitation 50% (water equivalent of 1.76 mm).

Thursday
Scattered Clouds. High: 20 °C . Wind East 18 km/h .

Thursday Night
Scattered Clouds. Low: 13 °C . Wind East 14 km/h .

Friday
Scattered Clouds. High: 21 °C . Wind East 14 km/h .

Friday Night
Clear. Low: 10 °C . Wind ESE 14 km/h . Windchill: 9

Extended Forecast

Updated: 7:00 PM IST on June 21, 2009

Wednesday Night
Partly Cloudy. Low: 12 °C . Wind SE 14 km/h .

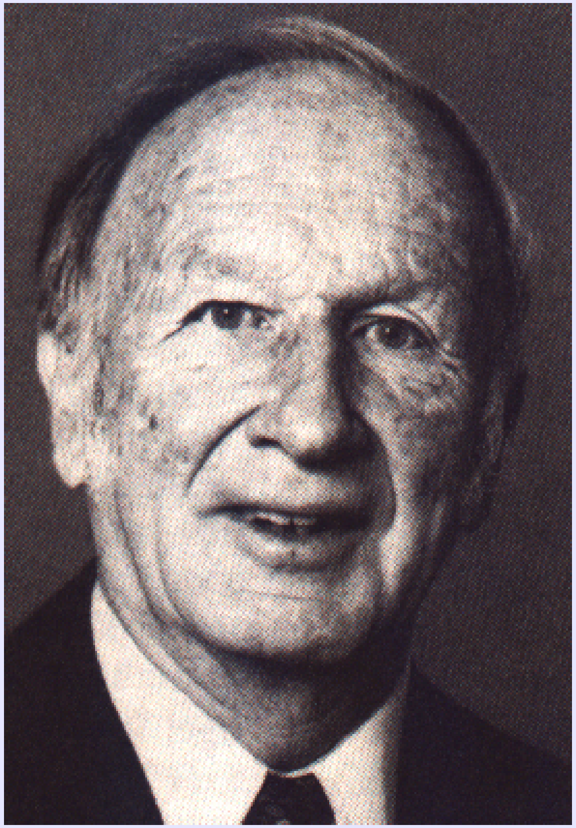
Thursday
Chance of Rain. Scattered Clouds. High: 21 °C . Wind ESE 18 km/h . Chance of precipitation 20% (trace amounts).

Thursday Night
Chance of Rain. Scattered Clouds. Low: 11 °C . Wind ESE 14 km/h . Chance of precipitation 20% (trace amounts).

Friday
Clear. High: 21 °C . Wind East 10 km/h .

Friday Night
Chance of Rain. Scattered Clouds. Low: 13 °C . Wind SE 10 km/h . Chance of precipitation 40% (water equivalent of 1.39 mm).

Edward Lorenz



Edward Lorenz 1917-2008

- during the 1950s became skeptical of the appropriateness of the mathematical models used in meteorology
- in 1963 published the famous paper: *Deterministic Nonperiodic Flow*
- the Lorenz equations:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = Rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

Outline

- **Dynamical systems**
- **Rigorous numerics of dynamical systems**
- **Homological invariants of dynamical systems**
- **Computing homological invariants**
- **Homology algorithms for subsets of $\mathbb{R}^d$**
- **Homology algorithms for maps of subsets of $\mathbb{R}^d$**
- **Applications**

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Dynamical systems

- X — a compact subset of $\mathbb{R}^d$ (in general: a topological space).
- $T \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{Z}^+\}$ — time (continuous if $T \in \{\mathbb{R}, \mathbb{R}^+\}$, discrete if $T \in \{\mathbb{Z}, \mathbb{Z}^+\}$)

A (semi)dynamical system is a continuous map

$$\varphi : X \times T \to X$$

such that for any $x \in X$ and $s, t \in T$

$$\varphi(\varphi(x,t),s) = \varphi(x, s+t)$$
$$\varphi(x,0) = x$$

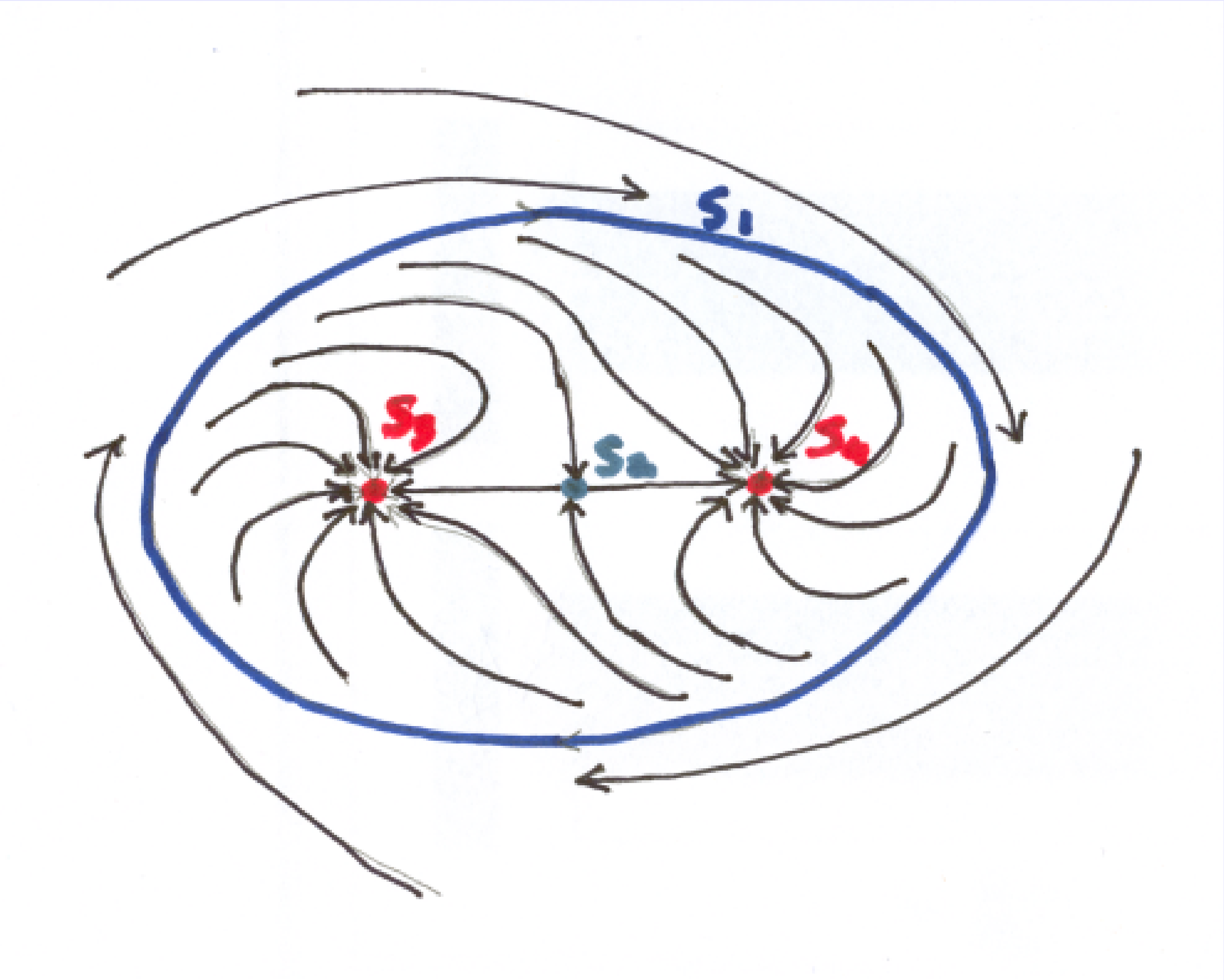
- $T = \mathbb{R}$ — a flow
- $T = \mathbb{Z}^+$ — a discrete semidynamical system (dsds)
- $\varphi_t : X \ni x \to \varphi(x,t) \in X$ — t-translation map
- $f := \varphi_1$ — the generator (for discrete time only) identified with sds

Invariant sets

- the trajectory (orbit) of $x \in X$
$$\varphi(x) := \{\, \varphi(x,t) \mid t \in T \,\}$$
- $x \in X$ is stationary iff $\varphi(x) = x$
- $x \in X$ is periodic iff there exists a $t \in T^+$ such that $\varphi(x,t) = x$
- the invariant part of $N \subset X$:
$$\operatorname{Inv}(N,\varphi) := \{\, x \in N \mid \varphi(x) \subset N \,\}$$
- $A \subset X$ is invariant if $\operatorname{Inv}(A,\varphi) = A$
- alpha and omega limit sets
$$\begin{aligned} \alpha(x) \; &:= \; \{\, y \in X \mid \exists t_n \to -\infty \text{ s.t. } y = \lim \varphi(x,t_n) \,\} \\ \omega(x) \; &:= \; \{\, y \in X \mid \exists t_n \to +\infty \text{ s.t. } y = \lim \varphi(x,t_n) \,\} \end{aligned}$$

Limit sets are invariant.

Invariant sets and limit sets



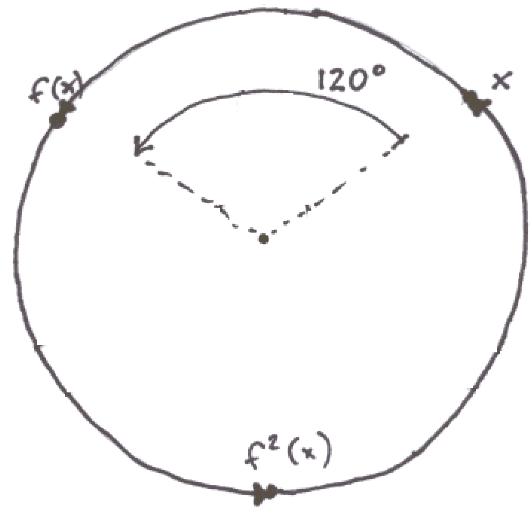
Some invariant sets and limit sets.

Asymptotic dynamics

- The main goal of the theory of dynamical systems is the understanding of the asymptotic behaviour of the trajectories, i.e. the number and structure of limit sets as well as their mutual relations
- Up to the half of the 20th century the dominating opinion was that the a limit set may be a stationary point or the trajectory of a periodic point.
- The computers significantly contributed to the realization that the asymptotic behaviour may be much more complicated (chaotic).

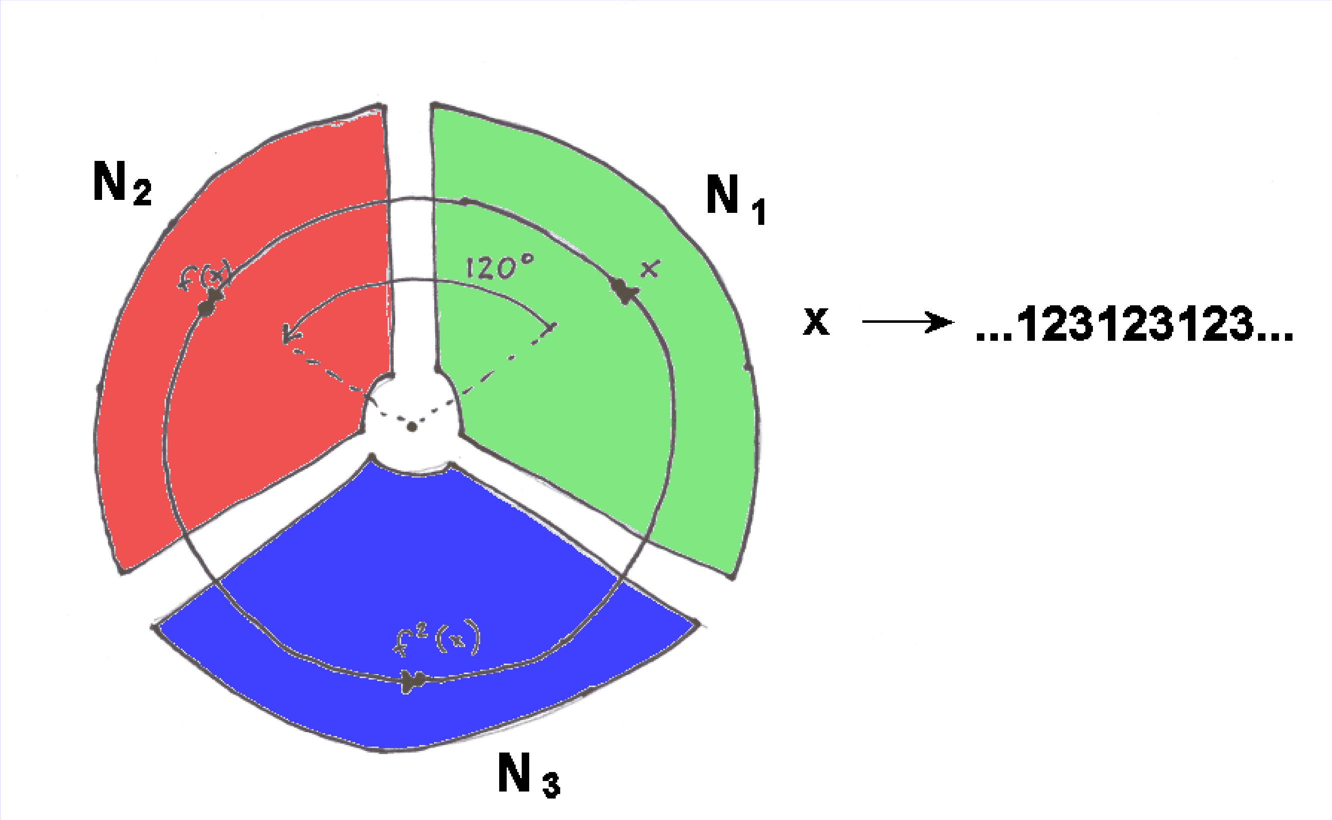
Main contributors to the discovery of deterministic chaos

- Henri Poincaré, 1890
- Mary Cartwright and John Littlewood, 1940's
- Andrey Kolmogorov and Yakov Sinai, 1950's
- Edward Lorenz, early 1960's
- Oleksandr Sharkovsky, 1964
- Stephen Smale, 1967
- Tien-Yien Li and James A. Yorke 1975



Rotation by 120 degrees

Symbolic dynamics



Mapping to sequences of symbols

Shift dynamics

- Consider
$$\Sigma_k := \{0, 1, 2, \ldots k-1\}^{\mathbb{Z}}$$
as a metric space with the metric
$$d(\alpha, \beta) := \sum_{i=-\infty}^{\infty} \frac{\delta_{\alpha(i),\beta(i)}}{2^{|i|}},$$
where δ_{mn} stands for the Kronecker delta.
- A full shift on k symbols is the discrete dynamical system generated on Σ_k by
$$\sigma : \Sigma_k \ni \alpha \to \sigma(\alpha) := (n \to \alpha(n-1)) \in \Sigma_k.$$

Features:

- Plenty of periodic points: Every finite sequence of symbols is in one-to-one correspondence with a periodic point of σ
- Sensitive dependence on initial conditions: trajectories diverge exponentially fast

Topological entropy

- For a finite open cover $\mathcal{C}$ of X let $\mathrm{card}_{\min} \mathcal{C}$ denote the minimal cardinality of all subfamilies of $\mathcal{C}$ which cover X
- For two families of sets $\mathcal{C}$ and $\mathcal{D}$ put

$$\mathcal{C} \vee \mathcal{D} := \{ C \cap D \mid C \in \mathcal{C}, D \in \mathcal{D}, C \cap D \neq \emptyset \}$$

Theorem. (Adler, Konheim, McAndrew, 1965) Let $f : X \to X$ be continuos and let $\mathcal{C}$ be a finite open cover of X. Then the limit

$$h(\mathcal{C}, f) := \lim_{n \to \infty} \frac{1}{n} \log \mathrm{card}_{\min}(\mathcal{C} \vee f^{-1}(\mathcal{C}) \vee \ldots \vee f^{-n+1}(\mathcal{C}))$$

exists.

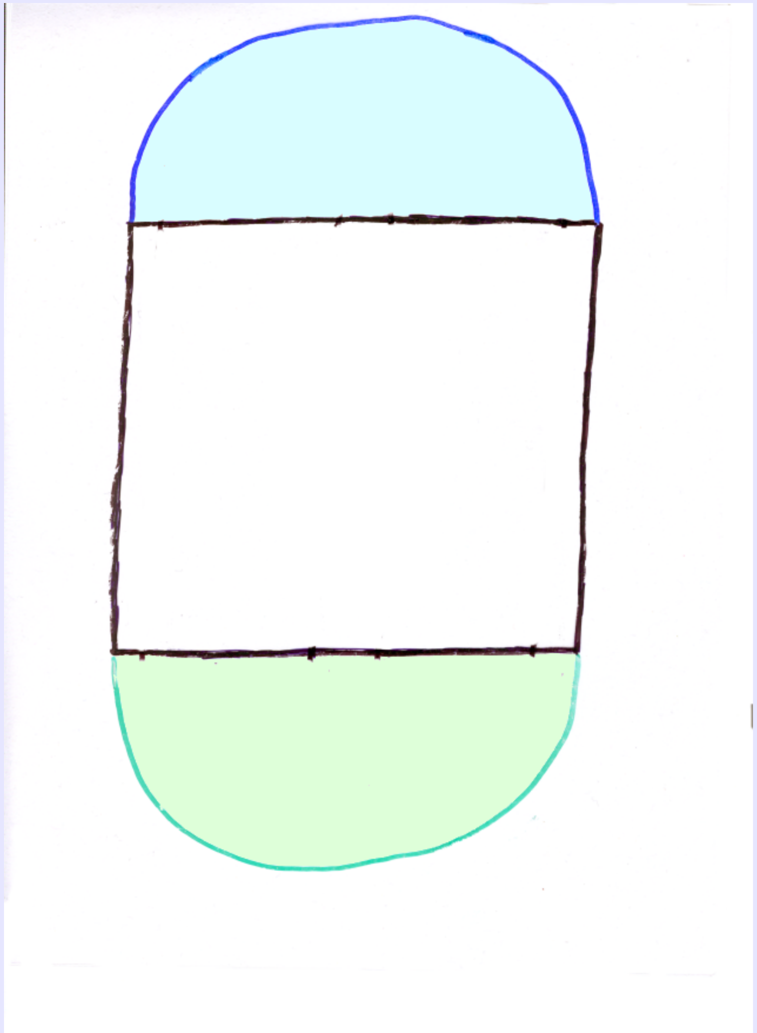
- the topological entropy:

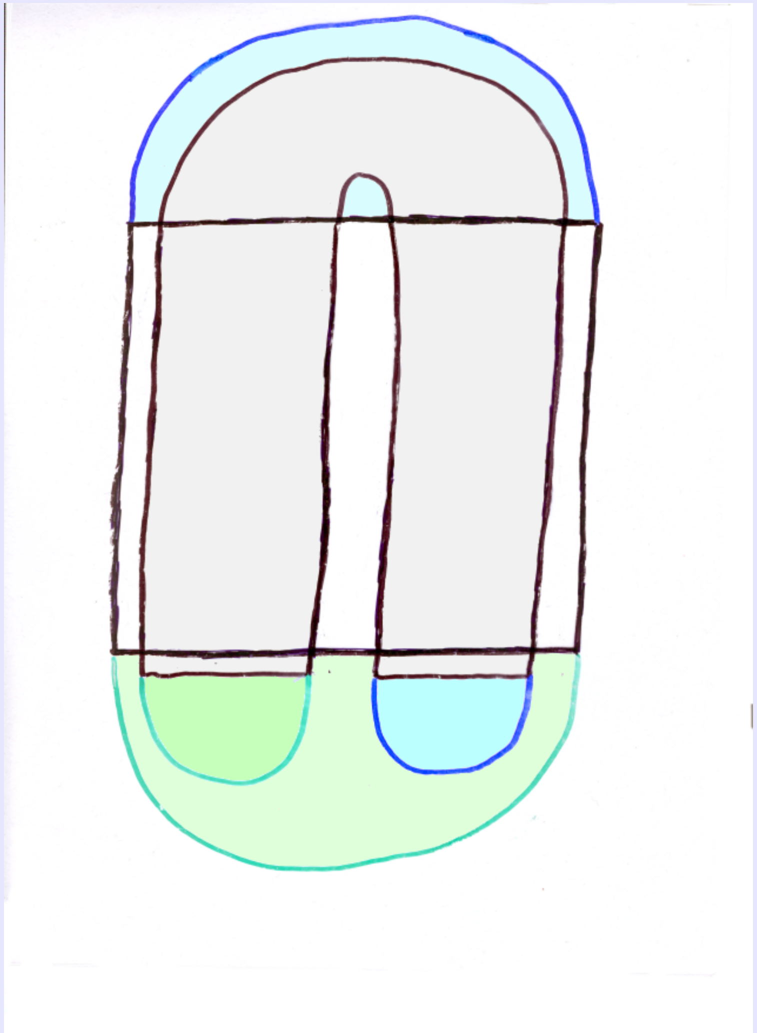
$$h(f) := \sup \{ h(\mathcal{C}, f) \mid \mathcal{C} \text{ a finite open cover of } X \}$$

Theorem. (Adler, Konheim, McAndrew, 1965)

- The entropy of an isometry is zero.
- The entropy of a full shift on k symbols is $\log k$.

Smale horseshoe (1967)





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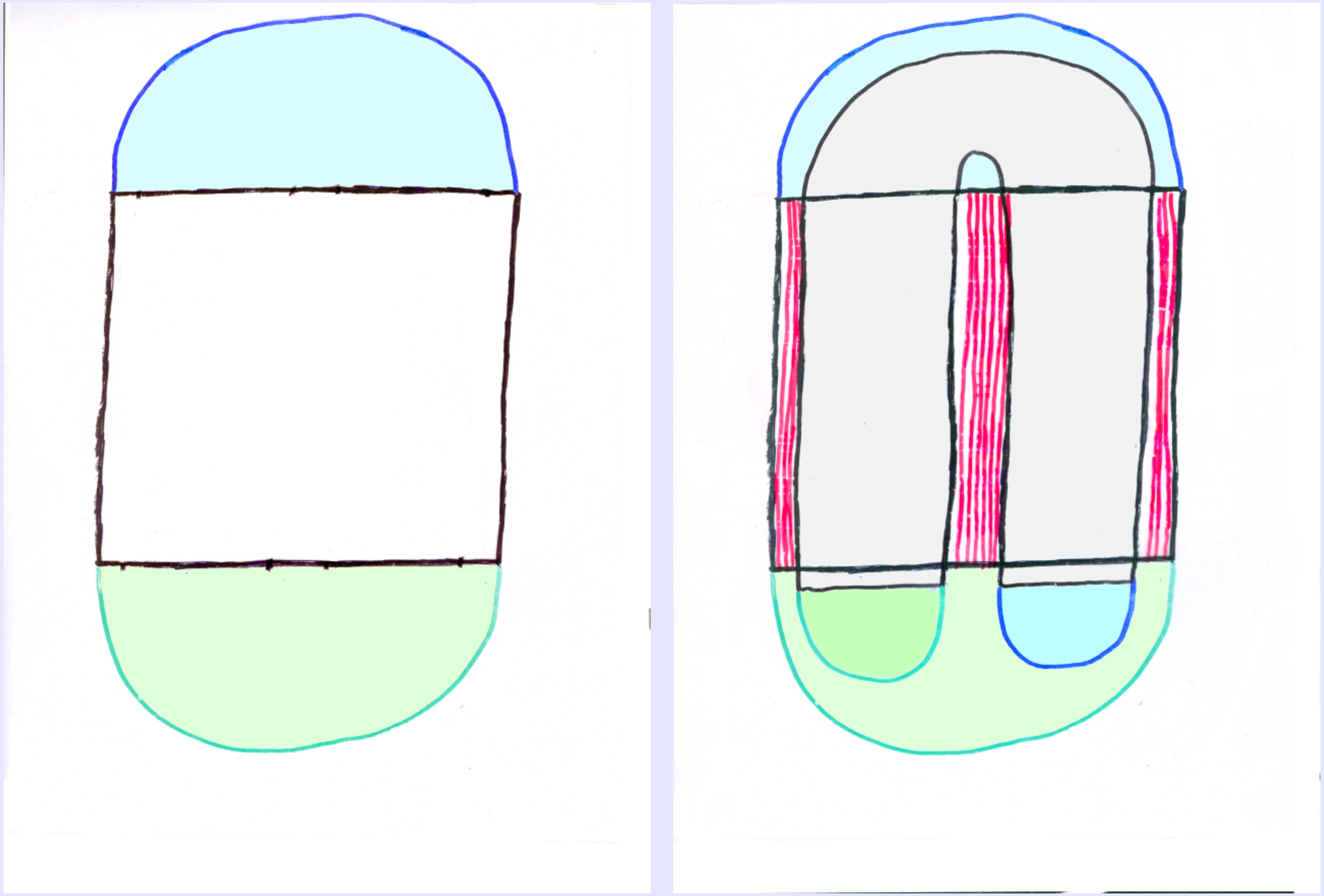

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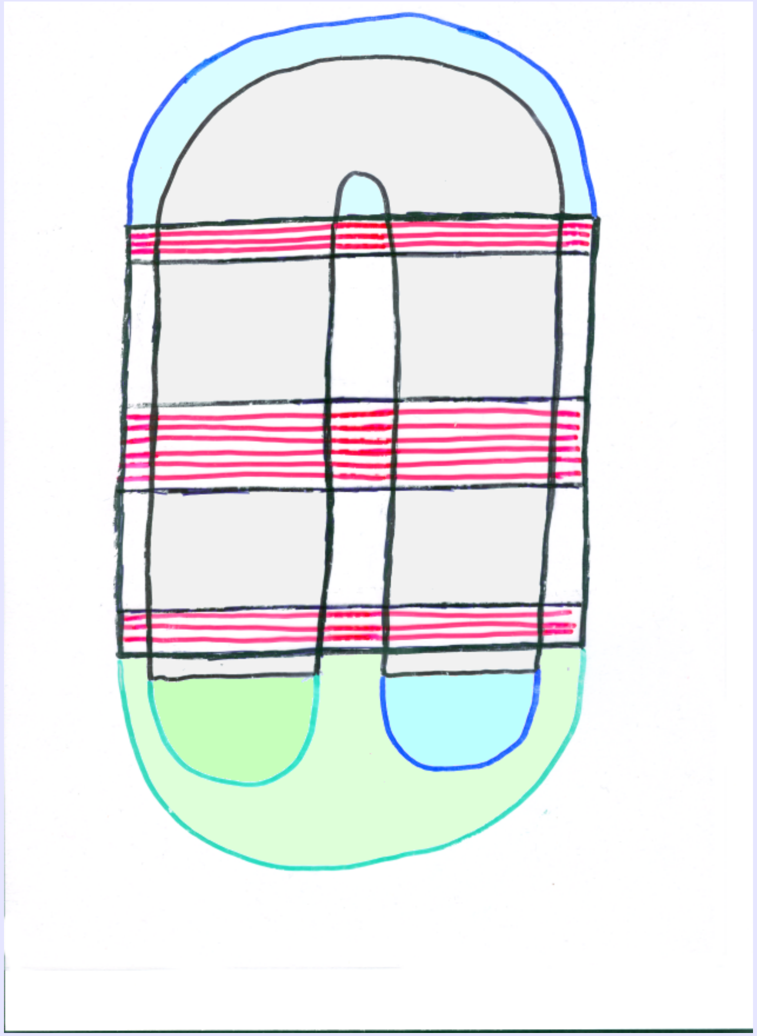

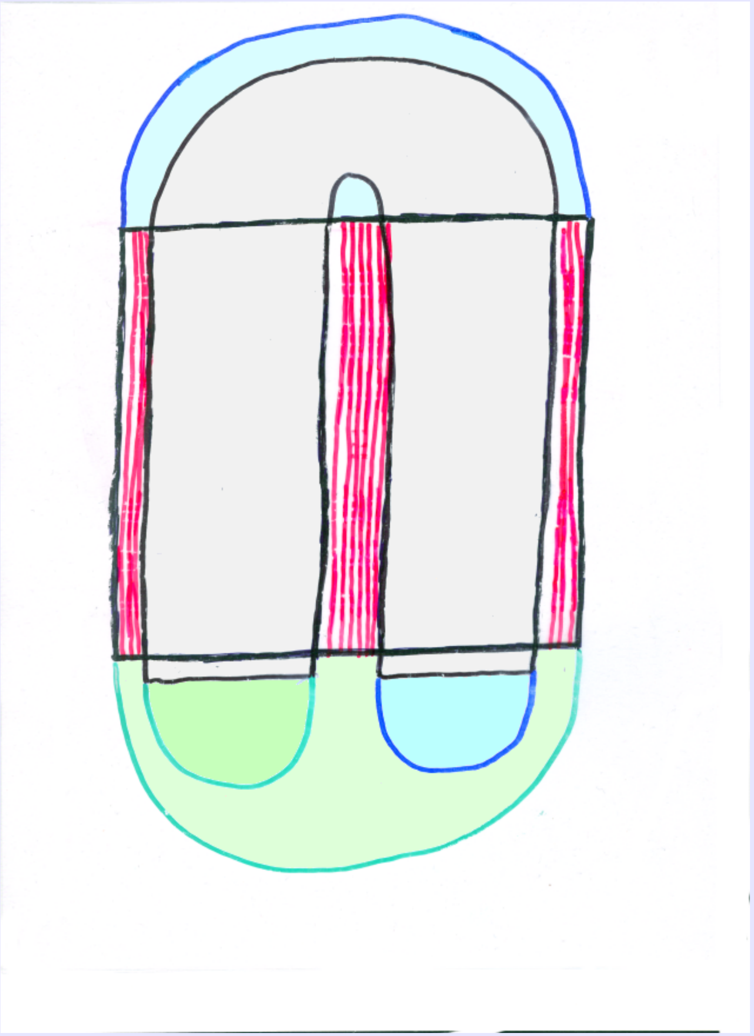

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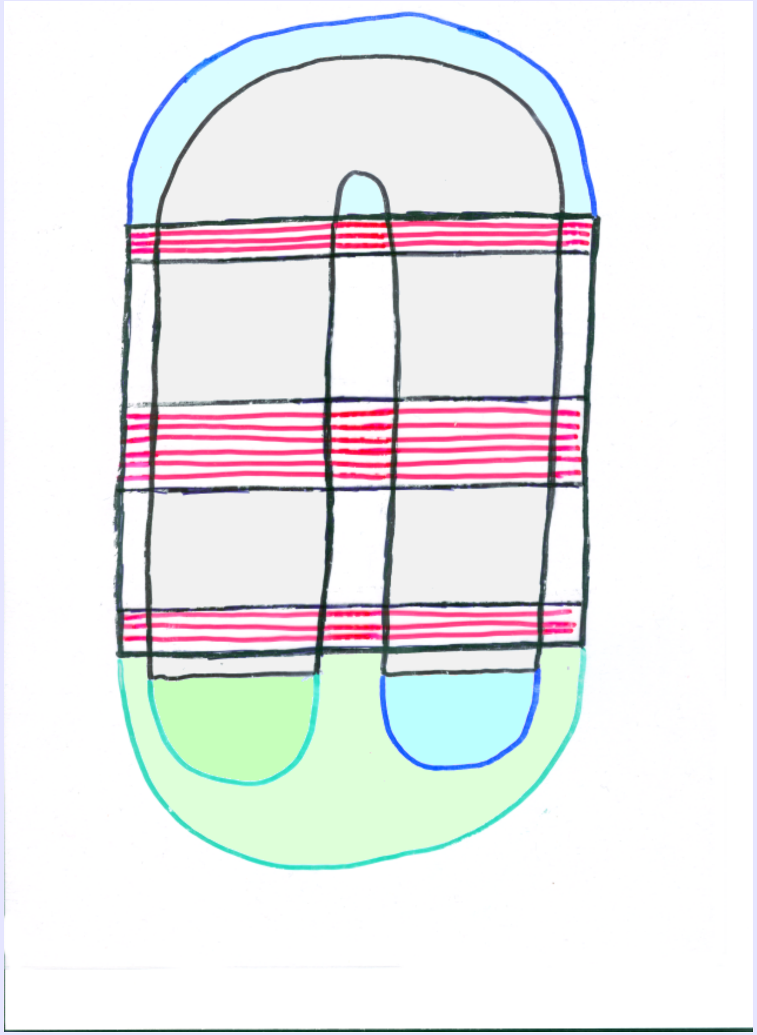

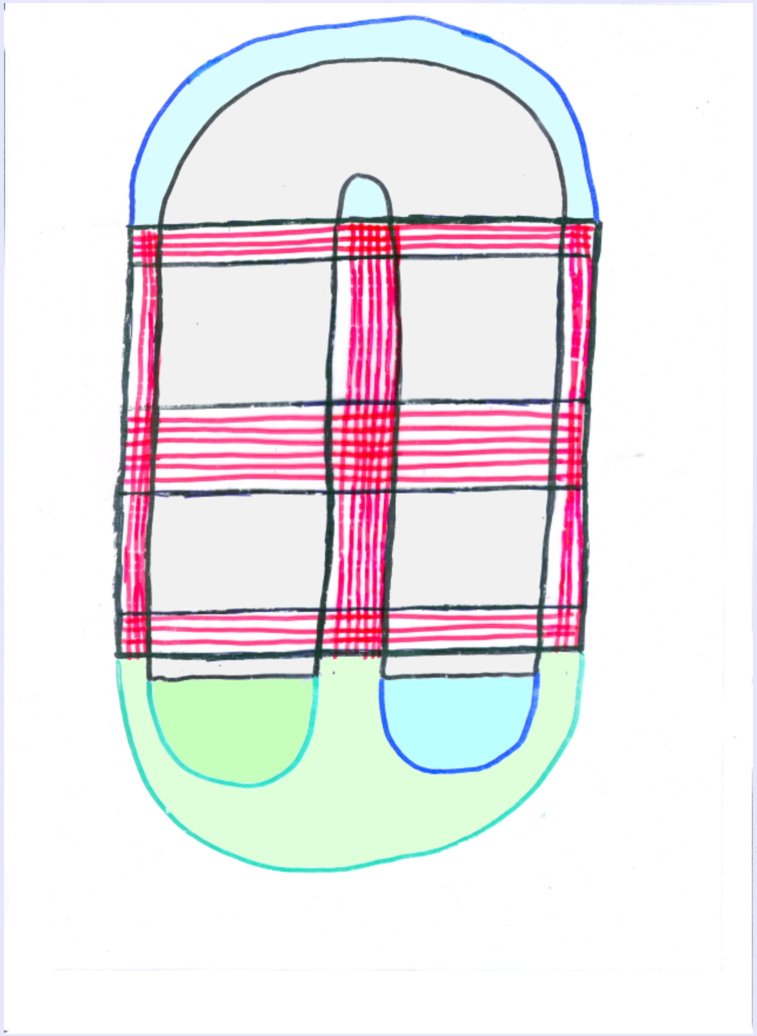

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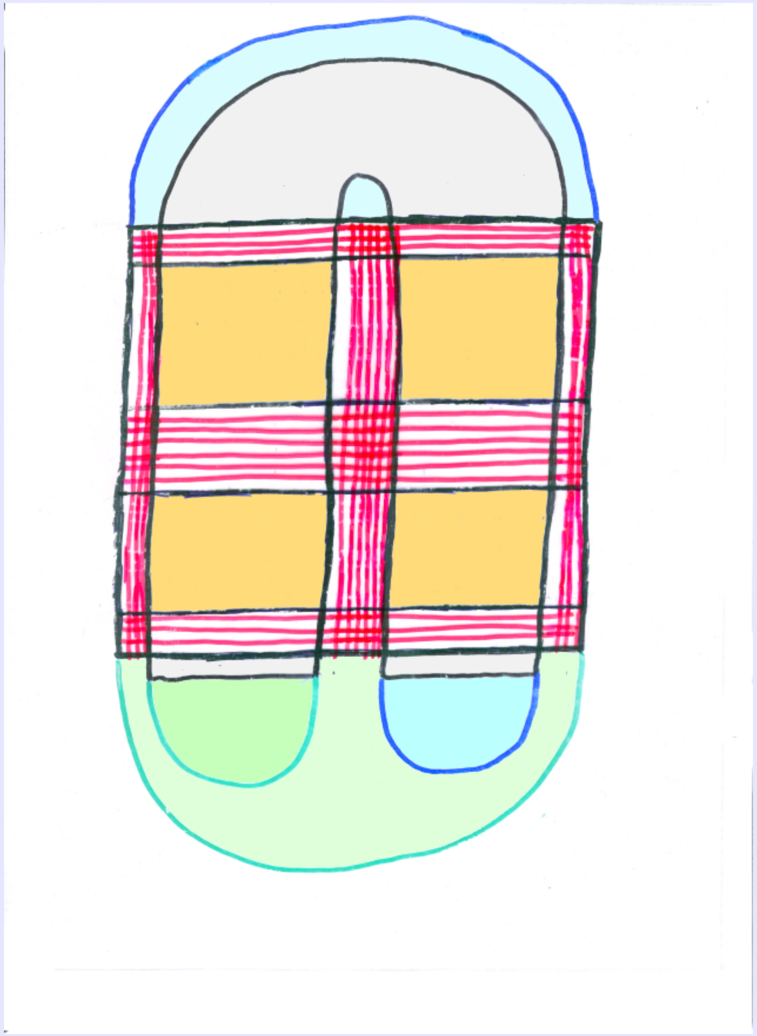

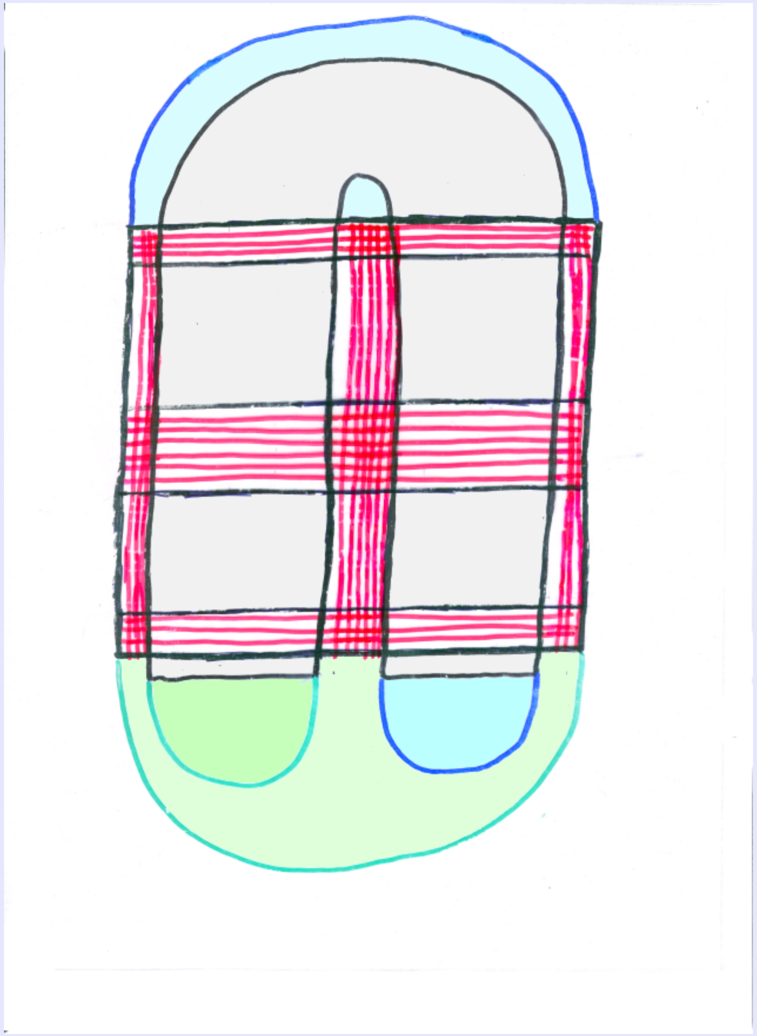

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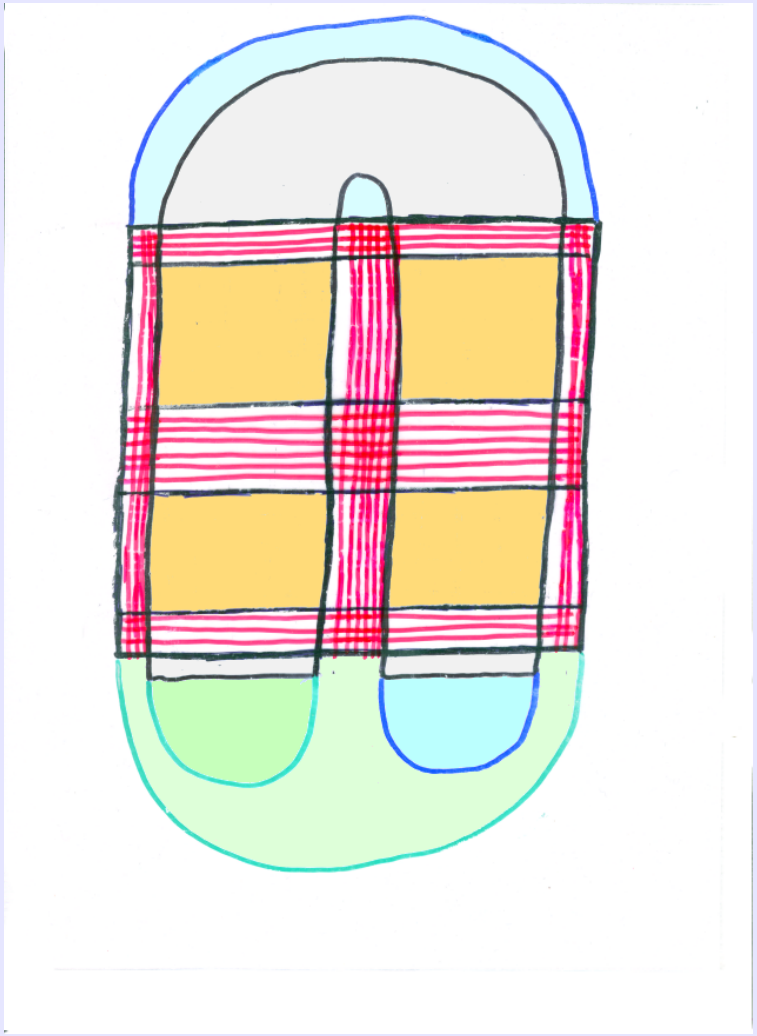

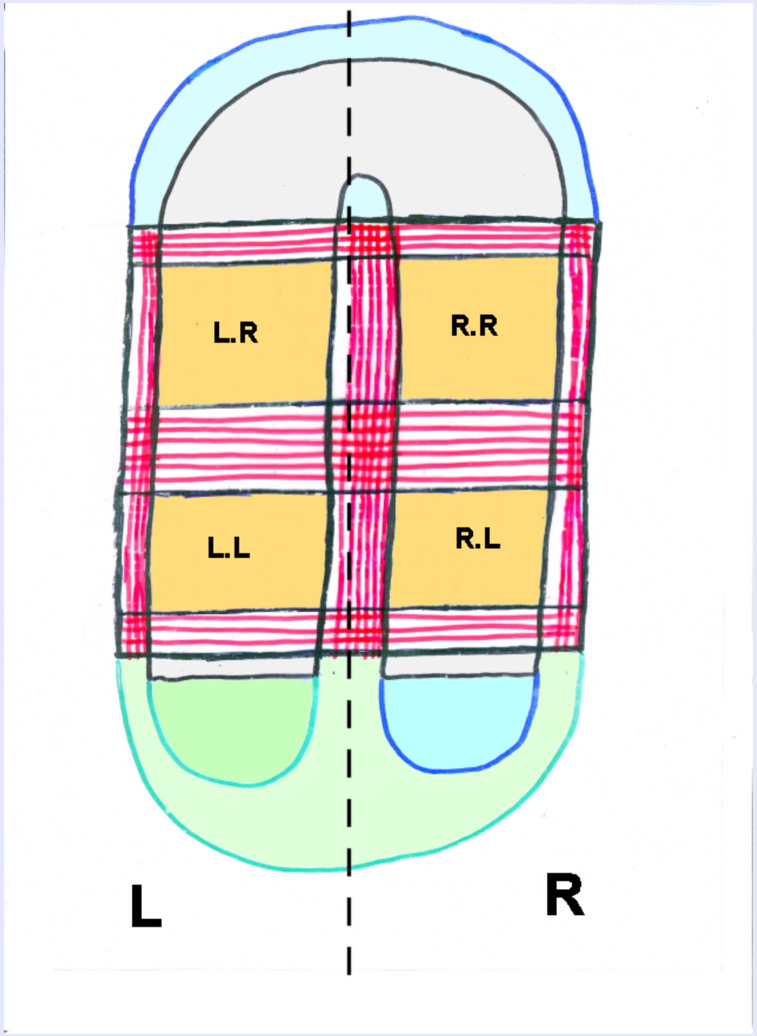

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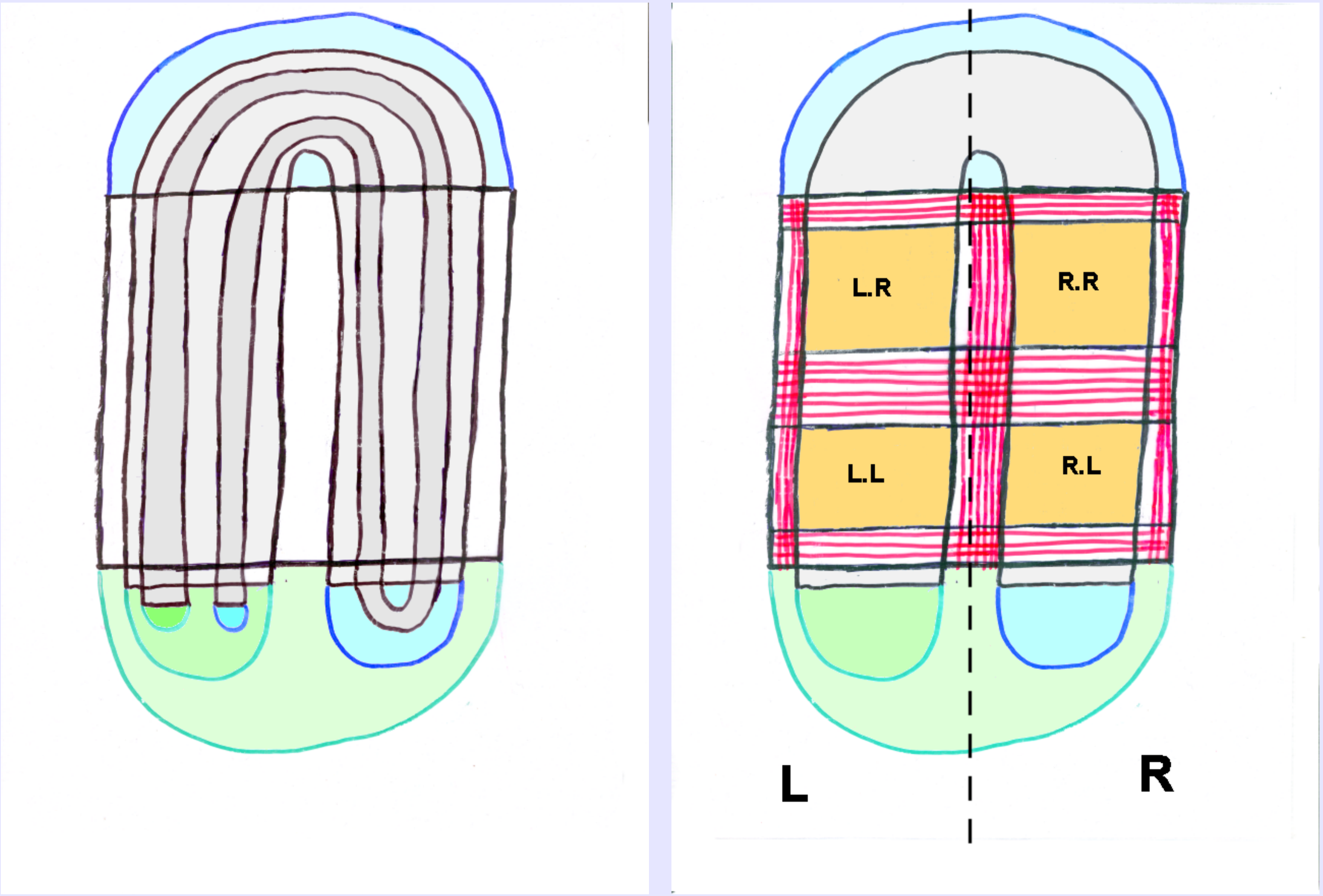

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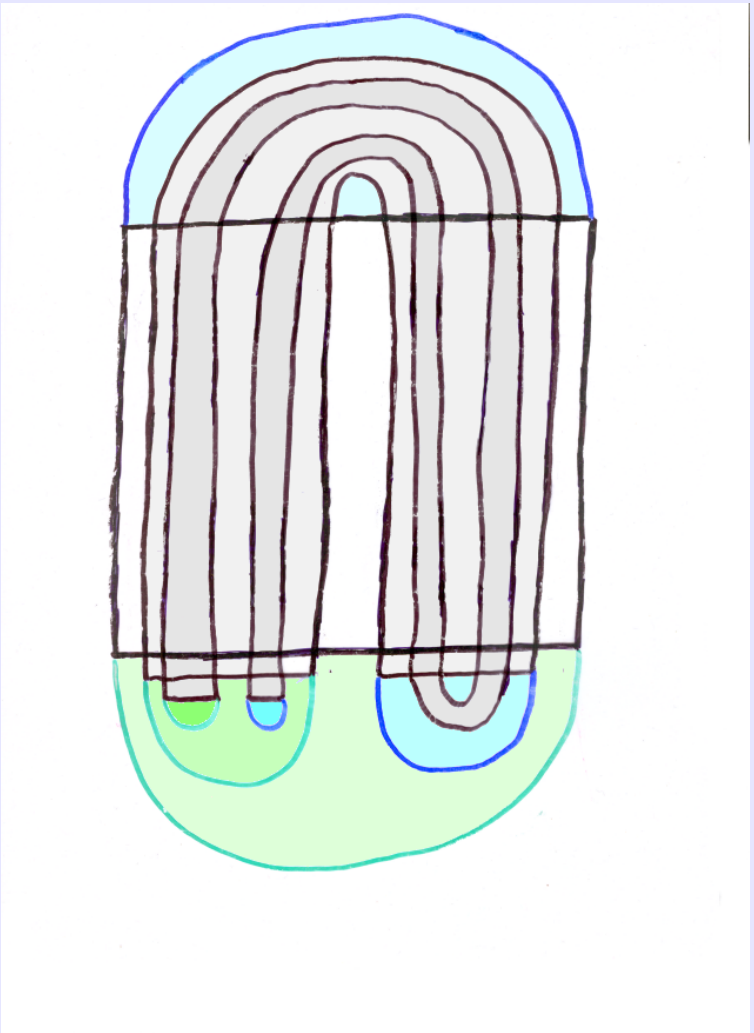
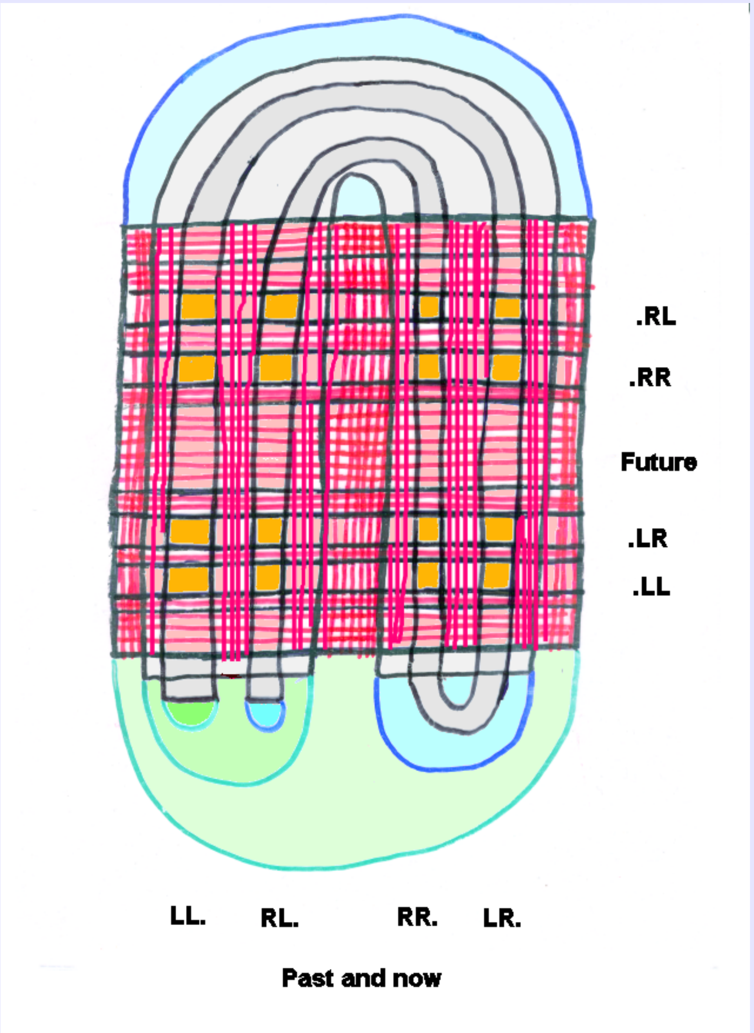
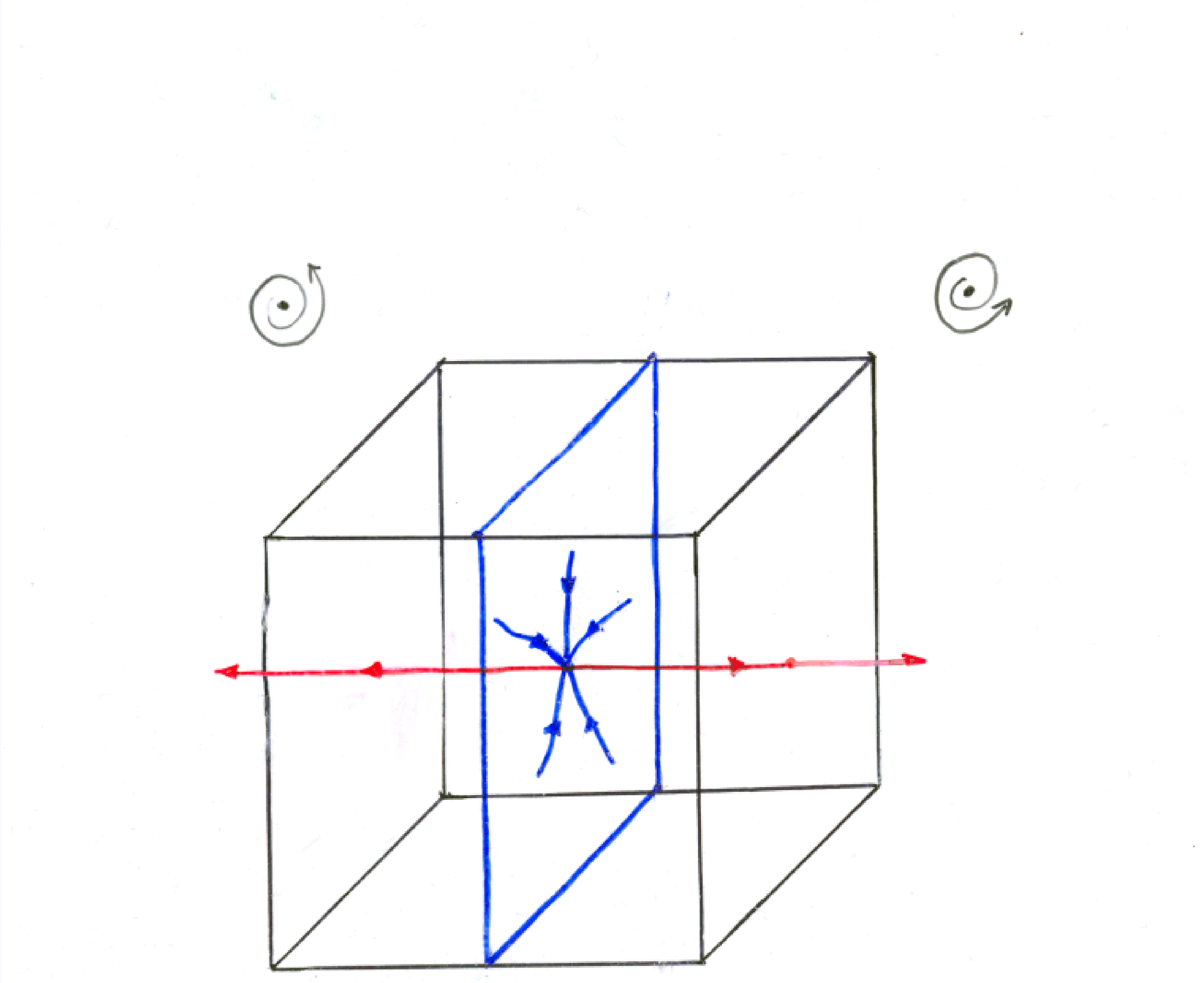




Smale horseshoe (1967)

Theorem. (Smale, 1967) Let N denote the square part of the domain of the horseshoe map h. Then there exists a homeomorphism $\rho : \mathrm{Inv}(N, h) \longrightarrow \Sigma_2$ such that $\sigma\rho = \rho h$.

Lorenz equations around the origin



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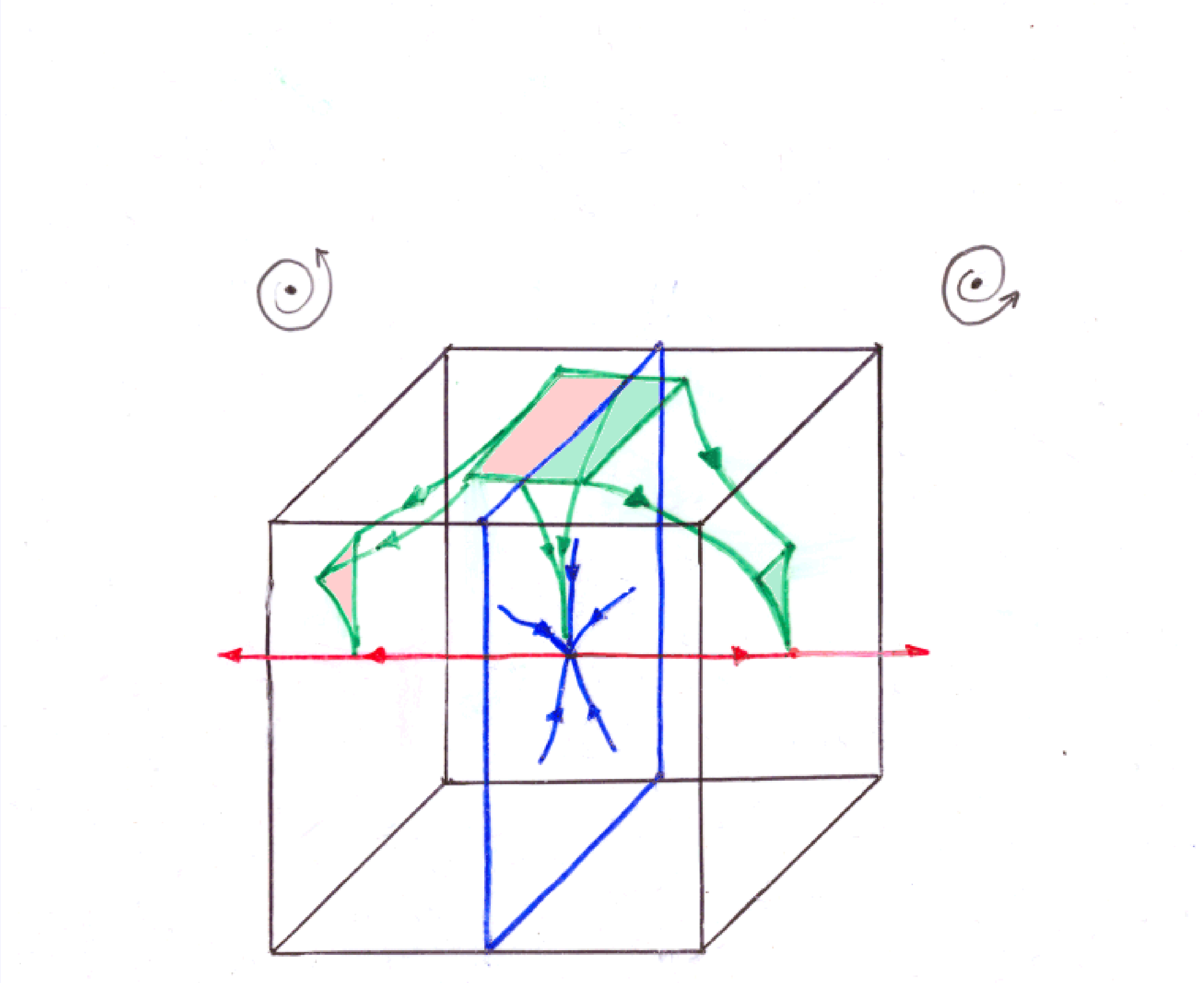

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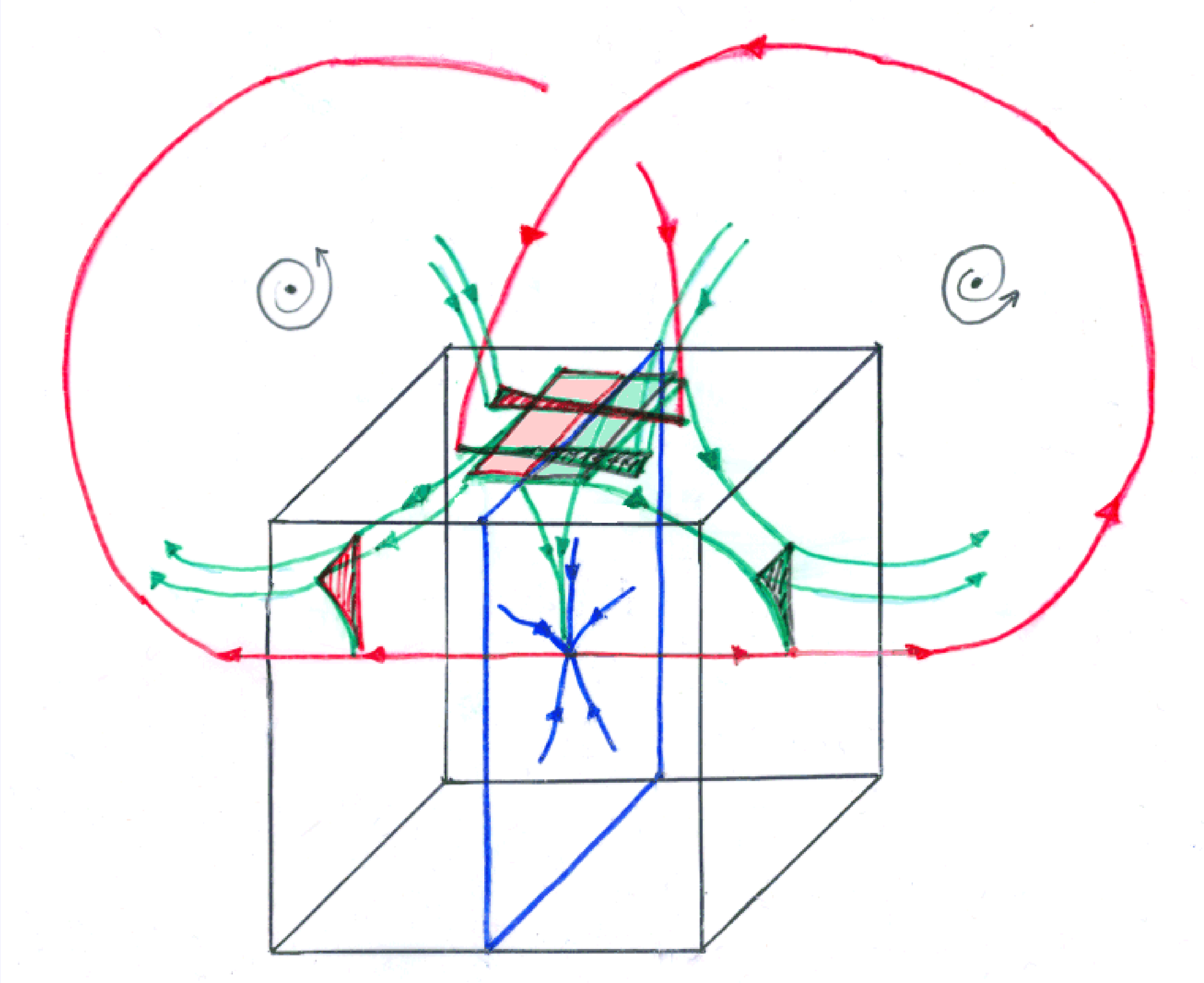
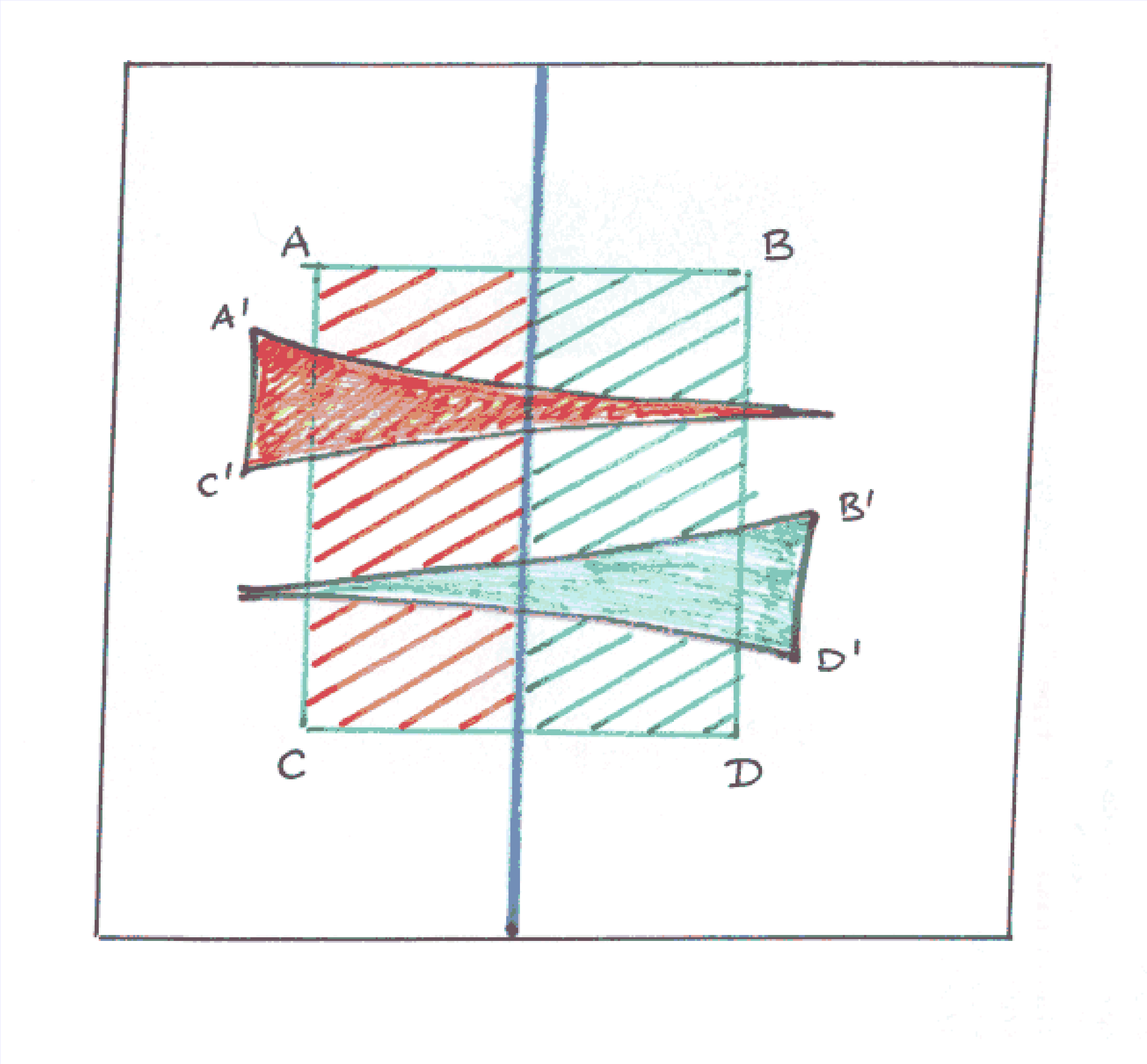


Poincaré map in the Lorenz equations



Problem

- Is horseshoe dynamics present in the Lorenz system?
- Or maybe it is only present in the numerical scheme?
- Or maybe the chaotic behaviour is only the consequence of the rounding errors?

References

- **E.N. Lorenz**, Deterministic Nonperiodic Flow, *Journal of the Atmospheric Science* (**1963**).
- **R.L. Adler, A.G. Konheim, M.H.McAndrew**, Topological entropy, *Transactions of the American Mathematical Society* (**1965**).
- **S. Smale**, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* (**1967**).
- **C. Sparrow**, The Lorenz Equations: Bifurcations, Chaos and Strange Attractors, *Springer-Verlag* (**1982**).