The metabelian BCH formula and compressed Drinfeld associators

Vitaliy Kurlin
Durham University, UK
http://maths.dur.ac.uk/∼dma0vk

On the crossroads

knots | **Lie algebras**

exponential equations

like $\exp(X)\exp(Y)=\exp(H)$

quantum algebra | **number theory**

The BCH formula

BCH = Baker + Campbell + Hausdorff

Problem : given $e^X e^Y = e^H$ find the solution H in terms of X, Y.

If X, Y commute then $H = X + Y$, otherwise $H = X + Y + [X, Y]/2 + \cdots$

A Lie algebra

Def : a Lie algebra L is a vector space with a linear operation $[,] : L \bigotimes L \rightarrow L$ satisfying the antisymmetry $[A, B] = -[B, A]$ and Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$

Examples of Lie algebras

$\mathbb{R}^n$ with the vector product (finite dim).

For the Lie agebra generated by X, Y, Z and $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$, we get $\log(e^X e^Y) = X + Y + Z/2$, where

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}, \log(1 + U) = -\sum_{n=1}^{\infty} \frac{(-U)^n}{n}.$$

A free Lie algebra

Def : a Lie algebra L is free if it has no additional relations apart from the antisymmetry and Jacobi identity.

L is the quotient of the algebra of non-associative monomials in X, Y over the ideal spanned by AA, $(AB)C + B(CA) + C(AB)$.

The Hausdorff series

The Hausdorff series $H = \log(e^X e^Y)$ lives in the free Lie algebra L generated by X, Y and starts as follows:

$$H = X + Y + \frac{[XY]}{2} + \frac{[X^2Y] - [YXY]}{12} + \dots,$$

$$[A_1 A_2 \dots A_n] = [A_1, [A_2 \dots, [A_{n-1}, A_n] \dots]]$$

The Bernoulli numbers

Def : the Bernoulli numbers are defined by the generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n = \frac{t}{e^t - 1}, \text{ for instance}$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0.$$

Property: $B_{2n+1} = 0$ for all $n \geq 1$.

A derivative in a Lie algebra

Def : the graded completion $\hat{L}$ of L is the algebra of infinite formal series of elements from L, e.g. the series $H \in \hat{L}$.

A derivative is a linear map $D : \hat{L} \to \hat{L}$ such that $D[A, B] = [DA, B] + [A, DB]$, the adjoint operator $\mathrm{ad}\, X(A) = [X, A]$.

Classical BCH formula

A recursive formula by Baker (1898), Campbell (1905), Hausdorff (1906).

Set $H_1 = X + \sum_{n=1}^{\infty} \frac{B_n}{n!}[Y^n X]$. Define

$D_Y(X) = 0$, $D_Y(Y) = H_1$. Then

$$H = \sum_{m=0}^{\infty} H_m,\ H_0 = Y,\ H_m = \frac{D_Y(H_{m-1})}{m}.$$

The Dynkin formula

The Hausdorff series $H = \log(e^X e^Y)$ is

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{\substack{p_i + q_i > 0 \\ p_i, q_i \geq 0}} \frac{[X^{p_1} Y^{q_1} \dots X^{p_m} Y^{q_m}]}{p_1! q_1! \dots \sum_{i=1}^{m} (p_i + q_i)}$$

The terms of the above series are linearly dependent: $[XY] = -[X^0 YXY^0]$

In associative monomials

Express $H = \log(e^X e^Y)$ as follows:

$$\sum c_x(s_1, \ldots, s_m) X^{s_1} Y^{s_2} \cdots (X \vee Y)^{s_m} +$$
$$\sum c_y(s_1, \ldots, s_m) Y^{s_1} X^{s_2} \cdots (Y \vee X)^{s_m},$$

$(X \vee Y)^{s_m}$ is X^{s_m} or Y^{s_m} for odd/even m

The Goldberg formula

$$\sum_{s_1,\dots,s_m} c_x(s_1,\dots,s_m) Z_1^{s_1}\cdots Z_m^{s_m} =$$

$$\sum_{i=1}^m Z_i e^{m' Z_i} \prod_{j\neq i} \frac{e^{Z_j}-1}{e^{Z_i}-e^{Z_j}} \text{ and}$$

$$c_y(s_1,\dots,s_m) = (-1)^{n-1} c_x(s_1,\dots,s_m),$$

where $m' = [m/2]$, $n = \sum_{i=1}^m s_i$.

No repetitions, but not in a linear basis.

Exp infinite linear basis

Hall's basis of the free Lie algebra L generated by X, Y has $l(m)$ elements of degree m, where $\sum_{m|n} ml(m) = 2^n$, $n \geq 1$,

e.g. $l(1) = 2 : X, Y$, $l(2) = 1$: $[X, Y]$, $l(3) = 2$, $l(4) = 3$, $l(5) = 6$, $l(6) = 9$.

The asymptotics of $l(m)$ is exponential.

Growing free basis

$$
\begin{tikzcd}
& X \arrow[dr] & & Y \arrow[dl] & & \\
& & {[X,Y]} \arrow[dl] \arrow[drr] & & & \\
& {[XXY]} \arrow[dl] \arrow[dr] & & & {[YXY]} \arrow[dl] \arrow[dr] & \\
{[XXXY]} \arrow[d] \arrow[dr] & & {[YXXY]} \arrow[r, phantom, "="] \arrow[dd] & {[XYXY]} \arrow[dd] & & {[YYXY]} \arrow[dl] \arrow[d] \\
{[X^4Y]} & {[YX^3Y]} & & & {[XY^2XY]} & {[Y^3XY]} \\
& & {[X^2YXY]} & {[YXYXY]} & &
\end{tikzcd}
$$

The metabelian quotient

Def : for a Lie algebra L the metabelian quotient is $\bar{L} = L/[[L, L], [L, L]]$.

Then $[XYX^2Y] = [YXX^2Y]$ in $\bar{L}$ since $[XYX^2Y] - [YXX^2Y] = [[X, Y], [X^2Y]]$, i.e. X, Y 'commute' in front of $[\cdot, \cdot]$.

The metabelian basis

$$\begin{tikzcd}
 & & X \arrow[dr] & & Y \arrow[dl] & \\
 & & & {[X,Y]} \arrow[dl] \arrow[dr] & & \\
 & & {[XXY]} \arrow[dl] \arrow[dr] & & {[YXY]} \arrow[dl] \arrow[dr] & \\
 & {[XXXY]} \arrow[d] \arrow[dr] & & {[YXXY]=[XYXY]} & & {[YYXY]} \arrow[dl] \arrow[d] \\
 & {[X^4Y]} & {[YX^3Y]} \arrow[d, equal] & & {[XY^2XY]} \arrow[d, equal] & {[Y^3XY]} \\
 & & {[X^2YXY]} & & {[YXYXY]} &
\end{tikzcd}$$

Adjoint operators x, y

The linear basis of the metabelian quotient $\bar{L}$ consists of X, Y and $[X^k Y^l XY] = x^k y^l [XY]$, $k, l \geq 0$, where $x = \mathrm{ad}\, X$, $y = \mathrm{ad}\ Y$ commute on $[L, L]$, i.e. any series in elements of $[L, L]$ is a commutative series in x, y applied to $[X, Y]$, $H_1 = X - \sum_{n=1}^{\infty} B_n y^{n-1} [XY]/n!$

Metabelian BCH formula

Th (K, 2005) : let L be the free Lie algebra generated by X, Y. Under $L \to \bar{L} = L/[[L,L],[L,L]]$ the series $H = \log(e^X e^Y)$ maps to $\bar{H} =$

$$X + Y + \frac{1}{y}\left(1 - \frac{e^x - 1}{x} \cdot \frac{x+y}{e^{x+y} - 1}\right)[XY]$$

$\bar{H}$ is expressed in a linear basis of $\bar{L}$

Ideas of proof

2005 : induction using generalised Bernoulli numbers with symmetries.

2007 : rewriting generating functions of Goldberg's coefficients of $X^r Y^s$.

Applications : solving equations modulo commutators of commutators

Zassenhaus equation

$e^{X+Y} = e^X e^Y e^{C_2} e^{C_3} e^{C_4} \cdots$ has the solution modulo commutators of commutators : $\sum_{n=2}^{\infty} C_n = \frac{1}{x+y} \cdot$

$$\frac{e^{-y}-1}{y} \cdot \left(1 + \frac{e^{-x}-1}{x} \cdot \frac{y}{e^y - 1}\right) [XY].$$

Kashiwara-Vergne conj

It involves the commutator equation:
$\ln(e^X e^Y) - X - Y = [X, F] + [Y, G]$ for unknown F, G in the completion of the free Lie algebra L generated by X, Y.

Existence was proved in general by Alekseev and Meinrenken (2006).

The commutator equation

has solutions $G(X,Y) = F(-Y,-X)$,

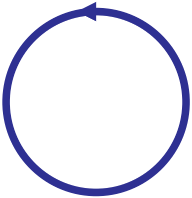
$F(X,Y) = aX + Y/4 + f(x,y)[XY]$,

$$\frac{1}{y(x-y)} - \frac{1}{4x} - \frac{e^x - 1}{x} \cdot \frac{x+y}{e^{x+y}-1} \cdot$$

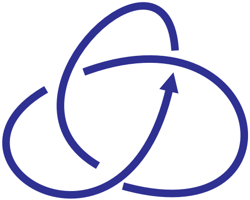
$$\cdot \frac{(x+y)e^y + 3x - y}{4xy(x-y)} + yg(x,y) = f(x,y),$$

g satisfies $g(x,y) = -g(-y,-x)$.

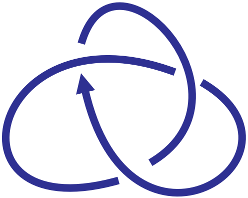
Knots and links



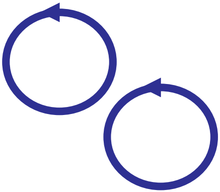
unknot



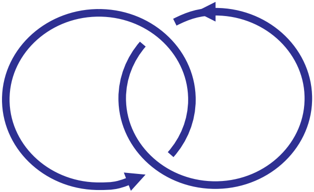
left trefoil



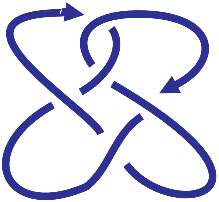
right trefoil



unlink



Hopf link



Whitehead

The Kontsevich integral

It is a powerful invariant of knots and can be computed from a plane diagram using a Drinfeld associator, whose logarithm lives in the Lie algebra L_3 generated by a, b, c modulo $[a, b] = [b, c] = [c, a]$.

A Drinfled associator

has the logarithm $\varphi(a,b) \in \hat{L}_3$ satisfying

symmetry: $\varphi(a,b) + \varphi(b,a) = 0$

hexagon: $e^{b+c} = e^{\varphi(c,a)} e^{c} e^{\varphi(b,c)} e^{b} e^{\varphi(a,b)}$

pentagon: $e^{\varphi(b,u)} e^{\varphi(a+c,d+u)} e^{\varphi(a,b)} =$

$$= e^{\varphi(a,b+d)} e^{\varphi(b+c,u)}$$

Multiple zeta values

Existence of a solution of the pentagon, hexagon was proved by Drinfeld (1990)

One transcendental solution was expressed in terms of multiple zeta values $\zeta(n_1, \ldots, n_s) = \sum_{k_1 > \ldots > k_s} \frac{1}{k_1^{n_1} \ldots k_s^{n_s}}$

Classical zeta values

The transcendental solution modulo commutators of commutators involves only zeta values $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$.

Hot problem: odd zeta values are polynomially independent?

Compressed associators

The metabelian BCH formula allows us to solve the hexagon and pentagon modulo commutators of commutators.

They do not contain polynomial relations between odd zeta values: $\zeta(3)^r \zeta(5)^s \ldots$ can be chosen as free parameters.

Hot problem: find a rational associator.

Strange triple symmetry

The homogeneous polynomials

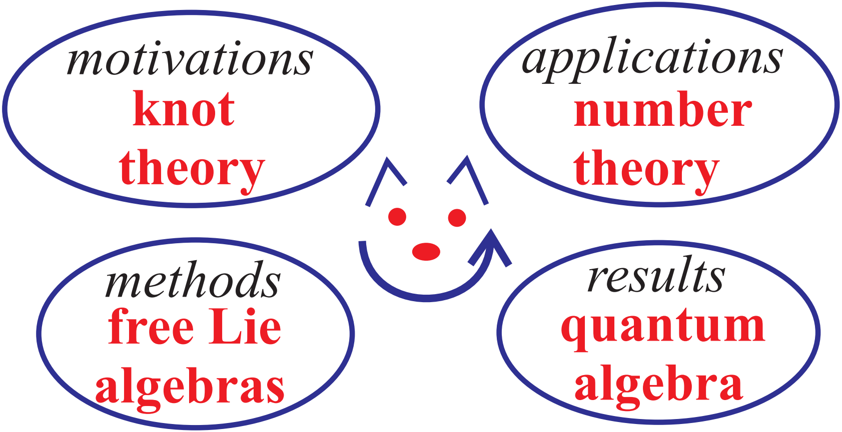
$$F_n(x, y) = F_n(y, x) = F_n(x, -x - y)$$

played a key role in the hexagon.

F_{2n} is a linear combination of

$$x^{2k} y^{2k} (x + y)^{2k} (x^2 + xy + y^2)^{n-3k}.$$

The research story



Problems and references

- write the BCH formula in a basis
- the compressed Kontsevich integral

- J Lie Theory **17** (2007), 525-538
- J Algebra **292** (2005), 184-242