The Cohomology Groups of a Quadratic Monomial Algebra

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Overview Of Talk

- Provide an overview of calculating the cohomology groups and cohomology algebra for an arbitrary associative algebra.

- Calculate the cohomology group for the class of Quadratic Monomial Algebras.

Theorem : For a given arbitrary associative $k-$algebra A, (k a field) the complex

$$\mathcal{B} : \cdots \xrightarrow{d_{n+1}} \mathbf{P}_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} \mathbf{P}_1 \xrightarrow{d_1} \mathbf{P}_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

is a projective resolution of A as an A^e-module, where

$$\mathbf{P}_0 = A \otimes_k A \quad \text{and} \quad \mathbf{P}_n = A \otimes_k A^{\otimes n} \otimes_k A.$$

The boundary d_n is given by

$$d_n (a_0 \otimes ... \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_{n+1}$$

and

$$\varepsilon (a \otimes b) := ab.$$

We apply $\mathrm{Hom}_{A^e}(\,\cdot\,,\, M\,)$, where M is an $A-$bimodule to

$$\cdots \xrightarrow{d_{n+1}} \mathbf{P}_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} \mathbf{P}_1 \xrightarrow{d_1} \mathbf{P}_0 \longrightarrow 0$$

to get the cochain complex:

$$0 \longrightarrow \mathrm{Hom}_{A^e}(\mathbf{P}_0, M) \xrightarrow{\delta^0} \mathrm{Hom}_{A^e}(\mathbf{P}_1, M) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-1}} \mathrm{Hom}_{A^e}(\mathbf{P}_n, M) \xrightarrow{\delta^n}$$

The nth Hochschild cohomology module of A with coeffients in M is

$$H^n(\,A,\,M\,) \cong \ker\delta^n/\mathrm{im}\,\delta^{n-1} \cong \mathrm{Ext}^n_{A^e}(\,A,\,M\,).$$

When $M = A$, we shall write $HH^n(A)$, in place of $H^n(A,\,A)$.

The Cohomology Algebra of A

We consider the graded vector space,

$$HH^*(A) = \bigoplus_{n \geq 0} HH^n(A) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_{A^e}(\,A,\,A\,)$$

and we define a multiplication on it, turning $HH^*(A)$ into a graded commutative algebra.

Any projective resolution $\mathbf{P}_*$ of A, contains all the necessary ingredients to construct a multiplication on $HH^*(A)$, where the product on $HH^*(A)$ is induced by a composition of chain maps on $\mathbf{P}_*$.

We may define a composition of cohomology classes $[\,f\,]$ and $[\,g\,]$ by:

$$[\,f\,] \cdot [\,g\,] := [\,f \circ \tilde{g}\,]$$

which makes $HH^*(A)$ into a graded commutative k-algebra.

- The composition product is well defined on $HH^*(A)$.

- The composition product is independent of the choice of resolution of A.

- The composition product is associative.

- The composition product is graded commutative.

Quivers, Quadratic Momomial Algebras

Definition : A *quiver* $\Delta = (\Delta_0, \Delta_1)$, is an oriented graph, where Δ_0 is the set of *vertices,* and Δ_1 is the set of *arrows* between the vertices. The origin and terminus of an arrow $a \in \Delta_1$, is denoted by $o(a)$ and $t(a)$ respectively.

A *path* α of length *n* in Δ, is an ordered sequence of arrows, $\alpha = a_1 \cdots a_n$, $a_i \in \Delta_1$ with $t(a_i) = o(a_{i+1})$ for $i = 1, \ldots, n-1$. The set of all paths of length n, is denoted Δ_n.

Let $k\Delta = \bigoplus_{i=0}^{\infty} k\Delta_i$ denote the $\mathbb{N}-$graded vector space over k, spanned by the set of all paths in Δ. We may endow $k\Delta$ with the structure of an algebra, where multiplication is given as the concatenation of paths.

In this talk, we investigate the class of *quadratic monomial* algebras. A quadratic monomial algebra A, is a quotient of a path algebra $k\Delta$. We write $A = k\Delta/I$, where $I = (\alpha_1, \ldots, \alpha_n)$ is a two sided homogenous ideal, generated by a set of paths of length two in Δ.

Lemma: If $A = k\Delta/I$ is a quadratic monomial algebra, then a minimal projective resolution of A given as a left A^e-module is

$$\mathbf{P}_i = A \otimes_{k\Delta_0} A^!_i \otimes_{k\Delta_0} A$$

and the A^e-linear differential is defined on the basis elements by

$$d_i(1 \otimes a_1 \cdots a_i \otimes 1) = a_1 \otimes a_2 \cdots a_i \otimes 1 + (-1)^i 1 \otimes a_1 \cdots a_{i-1} \otimes a_i$$

Proof: See Sköldberg.

Lemma : The map ϕ :$\mathrm{Hom}_{A^e}(\mathbf{P}_i, A) \longrightarrow \mathrm{Hom}_{k\Delta_0^e}(A_i^!, A)$

defined by

$$\phi(f)(a_1 \cdots a_i) := f(1 \otimes_{k\Delta_0} a_1 \cdots a_i \otimes_{k\Delta_0} 1)$$

is a chain map and an isomorphism of vector spaces for each i.

For $\alpha \in B(A^!)$ and $\beta \in B(A)$, a typical generating element in $\mathrm{Hom}_{k\Delta_0^e}(A_i^!, A)$ may be defined as the $k\Delta_0^e$-linear map

$$\alpha \stackrel{f'}{\longmapsto} \beta$$

and $\gamma \longmapsto 0$, for all other basis elements $\gamma \in B(A^!)$. We shall write (α, β) to denote the map f'.

A k-basis for $\mathrm{Hom}_{k\Delta_0^e}(A_i^!, A)$ is given by all (α, β), such that $o(\alpha) = o(\beta)$ and $t(\alpha) = t(\beta)$.

The differential δ on $\mathrm{Hom}_{k\Delta_0^e}(A_i^!, A)$. may be shown to have the form:

$$\delta(\alpha, \beta) = \sum_{a \in \Delta_1} (a\alpha, a\beta) + (-1)^{|\alpha|+1} \sum_{a \in \Delta_1} (\alpha a, \beta a)$$

The Hochschild cohomology groups are calculated using the procedure described next. The complex $\mathbf{K} = \mathrm{Hom}_{k\Delta_0^e}(A^!, A)$ is cut up into several smaller pieces by expressing it as a direct sum of several subcomplexes. Then the Hochschild cohomology is in turn calculated for each summand. In the following, the notation $\mathbf{K}_{(\alpha,\beta)}$ is used for the subcomplex of $\mathbf{K}$ spanned by the basis elements of the form $(\gamma\alpha\delta, \gamma\beta\delta)$ for some paths γ, δ in Δ.

Lemma : The complex **K**$=$ $\mathrm{Hom}_{k\Delta_0^e}(A^!, A)$, decomposes into the direct sum

$$\mathbf{K} \simeq \sum_{e\in\Delta_0} \mathbf{K}_{(e,e)} \oplus \coprod_{(\alpha,\beta)\in S_1} \mathbf{K}_{(\alpha,\beta)} \oplus \coprod_{\bar\alpha\in A^{!c}} \sum_{i=1}^{per\,\bar\alpha} \mathbf{K}_{(\alpha[i],o(\alpha[i]))}$$

$$\oplus \coprod_{\bar\beta\in A^c} \sum_{i=1}^{per\,\bar\beta} \mathbf{K}_{(o(\beta[i]),\beta[i]))} \oplus \coprod_{\alpha\in S_2} \mathbf{K}_{(\alpha,o(\alpha))} \oplus \coprod_{\beta\in S_3} \mathbf{K}_{(o(\beta),\beta)}$$

$$S_1 = \{ (a_1\cdots a_r,\, b_1\cdots b_s) \mid a_1 \neq b_1 \textit{ and } a_r \neq b_s,\ r\geq 1,\ s\geq 1\}$$

$$S_2 = \{ a_1\cdots a_r \in B(A^!) \mid t(a_r) = o(a_1),\ a_r a_1 \notin B(A^!),\ r\geq 2\}$$

$$S_3 = \{ b_1\cdots b_s \in B(A) \mid t(b_s) = o(b_1),\ b_s b_1 \notin B(A), s\geq 2\}$$

Proof : See Sköldberg.

Definition: Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an directed graph, where $\mathcal{V}$ and $\mathcal{E}$ denotes the vertex and edge set of Γ respectively.

Let $\tilde{\mathcal{C}}_*(\Gamma)$ denote the complex

$$\tilde{\mathcal{C}}_*(\Gamma): \quad 0 \longrightarrow \tilde{\mathcal{C}}_1(\Gamma) \xrightarrow{d_1} \tilde{\mathcal{C}}_0(\Gamma) \xrightarrow{d_0} \tilde{\mathcal{C}}_{-1}(\Gamma) \longrightarrow 0$$

where $\tilde{\mathcal{C}}_1(\Gamma)$ and $\tilde{\mathcal{C}}_0(\Gamma)$ are the $k-$spaces with basis the set of edges $e_i \in \mathcal{E}$, and the set of vertices $v_i \in \mathcal{V}$ respectively and $\tilde{\mathcal{C}}_{-1}(\Gamma)$ is the vector space with basis, the empty face $\emptyset$. Define

$$d_1(e_i) = t(e_i) - o(e_i) \quad \text{and} \quad d_0(v_i) = \emptyset, \quad \text{for each } i.$$

Let $\tilde{\mathcal{C}}^*(\Gamma) = \mathrm{Hom}_k(\tilde{\mathcal{C}}_*(\Gamma), k)$. In the forthcoming lemma, we describe the cohomology of a graph $\widetilde{H}^*(\Gamma) := H^*(\tilde{\mathcal{C}}^*(\Gamma))$.

Lemma : Let the number of vertices, the number of edges, and the number of connected components of a graph Γ, be denoted by $v(\Gamma)$, $e(\Gamma)$, and $c(\Gamma)$, respectively. The reduced cohomology of Γ is defined as $\widetilde{H}^*(\Gamma) := H^*(\tilde{\mathcal{C}}^*(\Gamma))$ where

$$\widetilde{H}^{-1}(\Gamma) = \begin{cases} 0 & \text{iff } \Gamma \text{ is nonempty,} \\ k & \text{otherwise.} \end{cases}$$

$$\widetilde{H}^0(\Gamma) = k^{c(\Gamma)-1}$$

$$\widetilde{H}^1(\Gamma) = k^{e(\Gamma)-v(\Gamma)+c(\Gamma)}$$