Polytopes, perturbations and cohomology computations

Scotland, October 2008

Graham Ellis
NUI Galway, Ireland

Problem: Compute

$$H_*(G, \mathbb{Z}) = H_*(BG, \mathbb{Z}) = \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$$

(where $BG = X/G$, any contractible space X with free G-acton).

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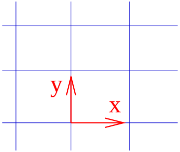
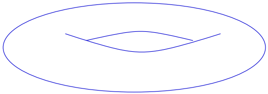
Illustration: $G = \langle x, y | xy = yx \rangle$, $X = \mathbb{R}^2$

X



X/G



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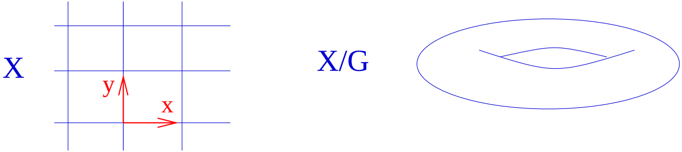

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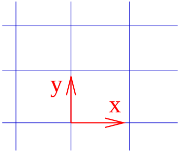

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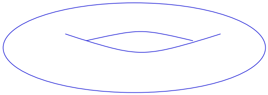

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$$H_n(G,\mathbb{Z}) = \frac{\ker(C_n(X) \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_{n-1}(X) \otimes_{\mathbb{Z}G} \mathbb{Z})}{\mathrm{Image}(C_{n+1}(X) \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_n(X) \otimes_{\mathbb{Z}G} \mathbb{Z}))}$$
$$= \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 1 \\ \mathbb{Z}, & n = 2 \end{cases}$$

COMPUTER EXAMPLE

(Number Crunching)

Low-dimensional homology of Mathieu groups

G	$H_2(G,\mathbb{Z})$	$H_3(G,\mathbb{Z})$	$H_4(G,\mathbb{Z})$	$H_5(G,\mathbb{Z})$
M_{11}	0	$\mathbb{Z}_8$	0	$\mathbb{Z}_2$
M_{12}	$\mathbb{Z}_2$	$\mathbb{Z}_6 \oplus \mathbb{Z}_8$	$\mathbb{Z}_3$	$(\mathbb{Z}_2)^3$
M_{21}	$\mathbb{Z}_4 \oplus \mathbb{Z}_{12}$	$\mathbb{Z}_5$	0	$(\mathbb{Z}_2)^4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7$
M_{22}	$\mathbb{Z}_{12}$	0	0	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_7$
M_{23}	0	0	0	$\mathbb{Z}_7$
M_{24}	0	$\mathbb{Z}_{12}$	$(\mathbb{Z}_2)^a$	$(\mathbb{Z}_2)^b \oplus (\mathbb{Z}_4)^c \oplus \mathbb{Z}_7$

$$0 \le a \le 32, 0 \le b \le 52, 0 \le c \le 1$$

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GAP commands

```
gap> GroupHomology(MathieuGroup(23),5);
[ 7 ]
```

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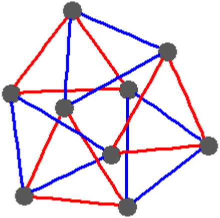
- $X^1_{(3)} =$ 

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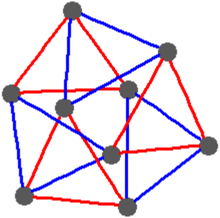
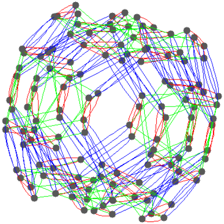
- $X^1_{(3)} =$  $X^1_{(2)} =$ 

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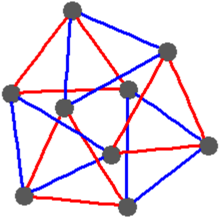
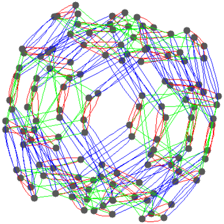
- $X^1_{(3)} =$  $X^1_{(2)} =$ 
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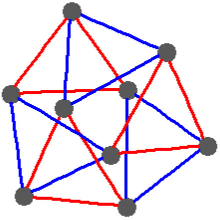
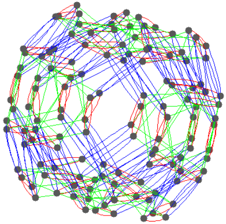
- $X^1_{(3)} =$  $X^1_{(2)} =$ 
- $C_*(X_{(p)})$ is a free $\mathbb{Z}P$-resolution of $\mathbb{Z}$.
- During construction of $X_{(p)}$ record an explicit contracting homotopy $h_* \colon C_*(X_{(p)}) \to C_{*+1}(X_{(p)})$ where, by definition

$$hd + dh = 1.$$

- $$\ker\left(\quad H_n(P,\mathbb{Z}) \twoheadrightarrow H_n(G,\mathbb{Z})_{(p)} \quad \right)$$
  described by homomorphisms
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R E S O L U T I O N S

(Polytopes and perturbations)

Easy example: The quaternion group of order eight

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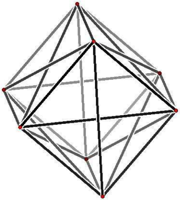

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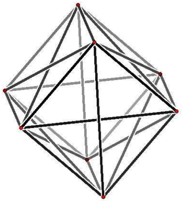

$C_*(P) : 0 \to C_4(P) \to C_3(P) \to C_2(P) \to C_1(P) \to C_0(P)$

exact sequence of $\mathbb{Z}G$-modules.

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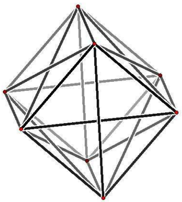

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$C_4(P) = \mathbb{Z}$ and $H_0(P) = \mathbb{Z} \Rightarrow$ periodic $\mathbb{Z}G$-resolution

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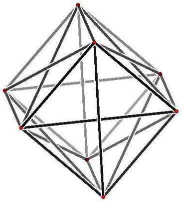

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In happens to be free!

```
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gap> TP:=TensorWithIntegers(P);;
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gap> PresentationOfResolution(P);
(relators := [ f2*f3^-1*f1^-1, f3*f2*f1^-1,
               f1*f2*f3, f1*f3^-1*f2 ] )
```

So

$$G =< i, j, k : ij = k, jk = i, ki = j, ikj = 1 > .$$

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IDEA:

- $M_{24} \leq S_{24}$ acts on $\mathbb{R}^{24}$. Compute

$$P = P_v(M_{24}) = \text{Convex Hull}(g.v : g \in M_{24})$$

- Determine face stabilizer subgroups.
- Combine $C_*(P)$ with free resolutions for stabilizer subgroups to get a free $\mathbb{Z}M_{24}$-resolution.

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If $v = (1, 2, 3, 4, 5, 0, \ldots, 0)$ then $P_v(M_{24}) = P_v(S_{24})$ and S_{24} is a finite reflection group!

Proposition

For a finite reflection group W generated by simple reflections $x_1, \cdots, x_n$, and for v not in a mirror, the polytope $P_v(W)$ has

$$k\text{-faces} \leftrightarrow \text{cosets of subgroups} < x_{i_1}, \cdots, x_{i_k} >$$

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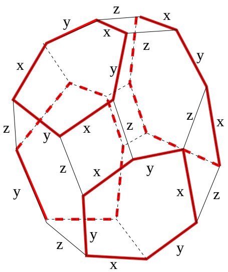
Example

S_4 with generators $x = (1,2), y = (2,3), z = (3,4)$



Observation

The length of those edges labelled by generator x in $P_v(W)$ decreases as the vector v is moved towards a mirror H_x corresponding to the reflection x.

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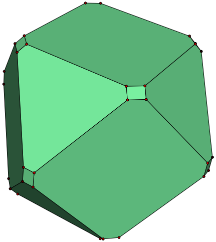
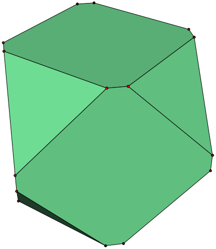
Example

$$W = \langle x, y, z : x^2 = y^2 = z^2 = (xy)^3 = (yz)^4 = (xz)^2 = 1 \rangle$$





Consequence

For W generated by $S = \{x_1, \ldots, x_n\}$, and for $D \subset S$, choose $v \in \cap_{x \in D} H_x$. Then $P_v(W)$ is an example of a *Wythoff polytope*.

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For $T \subseteq S$ set

$$T^{\perp} = \{y \in D \ : \ yx = xy \text{ for all } x \in T \cap (S \setminus D)\}.$$

Say $T \subseteq S$ is degenerate if there is a $y \in D \cap T$ such that $xy = yx$ for all $x \in T \cap (S \setminus D)$. Set $ND = \{T \subseteq S : T \text{ non} - \text{degenerate}\}$.

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Example

For $v = (1, 2, 3, 4, 5, 0, \dots, 0) \in \mathbb{R}^{24}$ and $P = P_v(M_{24})$ we can describe the non-free $\mathbb{Z}M_{24}$-resolution

$$\cdots \to C_3(P) \to C_2(P) \to C_1(P) \to C_0(P).$$

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Suppose that, for each p, we have a **free** $\mathbb{Z}G$-resolution of C_p

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There exists a free $\mathbb{Z}G$-resolution $R_*^G \to M$ with

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Can also be used to obtain a free $\mathbb{Z}G$-resolution

$$R_*^G = R_*^N \tilde{\otimes} R_*^{(G/N)}$$

where G is any group with normal subgroup N.

C O H O M O L O G Y R I N G S

(Gröbner Bases)

Cohomology $H^*(G, A)$ is a ring:

$$H^n(G, A) = \frac{\ker(Hom_{\mathbb{F}G}(R_n^G, A) \to Hom_{\mathbb{F}G}(R_{n+1}^G, A))}{\mathrm{Image}(Hom_{\mathbb{F}G}(R_{n-1}^G, A) \to Hom_{\mathbb{F}G}(R_n^G, A))}$$

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Easy example

$G_{2,4} = F/[F, [F, [F, [F, F]]]]$ where F is free of rank 2.

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```
gap> G:=NilpotentQuotient(FreeGroup(2),4);;
gap> R:=ResolutionNilpotentGroup(G,10);;
gap> ZG_Ranks_Of_R:=List([0..10],n->Dimension(R)(n));
[ 1, 8, 28, 56, 70, 56, 28, 8, 1, 0, 0 ]
gap> GeneratorsByDegree:=List([1..8],
                n->Size(IntegralRingGenerators(R,n)));
[ 2, 6, 10, 8, 4, 0, 0, 0 ]
```

Harder example: $H^*(Syl_2(M_{12}), \mathbb{F}_2)$

```
gap> P := SylowSubgroup(MathieuGroup(12), 2);;
gap> ModPCohomologyRingPresentation(P);
Graded algebra GF(2)[ x_1, x_2, x_3, x_4, x_5, x_6, x_7 ]
[ x_2*x_3, x_1*x_3, x_3*x_4, x_1^2*x_2+x_1*x_2^2+x_2^3+
  x_2*x_4+x_2*x_5, x_2^2*x_4+x_2*x_6, x_1^2*x_4+x_1*x_6+
  x_2*x_6+x_4^2+x_4*x_5, x_2^4+x_1*x_2*x_5+x_1*x_6+x_2*
  x_6+x_4*x_5, x_1^2*x_6+x_1*x_2*x_6+x_2^2*x_6+x_2*x_4*
  x_5+x_4*x_6, x_1*x_2^4+x_2^5+x_2^3*x_5+x_1*x_2*x_6+x_1*
  x_4*x_5+x_2*x_4*x_5+x_2*x_5^2+x_4*x_6, x_2^3*x_6+x_1*
  x_4*x_6+x_1*x_5*x_6+x_3^2*x_7+x_3*x_5*x_6+x_4^2*x_5+x_
  6^2 ] with indeterminate degrees [ 1, 1, 1, 2, 2, 3, 4 ]
gap> time;
11709
```

Analysis of GAP function

A free $\mathbb{F}G$-module $(\mathbb{F}G)^n$ is a vector space of dimension $n \times |G|$.
So linear algebra can be used to compute

$$H^*_N(G,\mathbb{F}) = H^*(G,\mathbb{F})/H^{*\geq N}(G,\mathbb{F}).$$

for any small G and $N \geq 1$.

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For sufficiently large N a presentation for $H^*(G, \mathbb{F})$ can be got from a presentation of $H^*_N(G, \mathbb{F})$.

Finding N for the quaternion group G

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- The central extension $1 \to C_2 \to G \to C_2 \times C_2 \to 1$ yields the LHS spectral sequence

$$E_2^* = H^*(C_2 \times C_2, \mathbb{F}) \otimes H^*(C_2, \mathbb{F}) = \mathbb{F}[x, y, z] \Longrightarrow H^*(G, \mathbb{F})$$

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- Our CTC Wall resolution defines the derivation $d_2 \colon E_2^* \to E_2^*$ by $d_2(x) = d_2(y) = 0$, $d_2(z) = x^2 + xy + y^2$.

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$$E_3^* = \ker(d_2)/\mathrm{image}(d_2) = \mathbb{F}[x, y, z^2]/ < x^2 + yy + y^2 >$$

- Using the CTC Wall resolution to obtain the differential on E_3^*, repeat to find

$$E_4^* = E_\infty^* = \mathbb{F}[x, y, z^2]/ < x^2 + xy + y^2,\ y^3 >$$

- Can set $N = 4$.

INFINITE GROUPS

Example 1

Does the presentation

$$G = \langle x, y, z : xyxyx = yxyxy, yzyz = zyzy, xzx = zxz \rangle$$

correspond to an aspherical 2-dimensional CW-space X?

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We can change the CW-decomposition on X by considering

$$G = \langle x, y, z, a, b, c \ : \ xy = a,\ yz = b,\ zx = c, \\ a^2x = ya^2,\ b^2 = zby,\ cz = xc \rangle.$$

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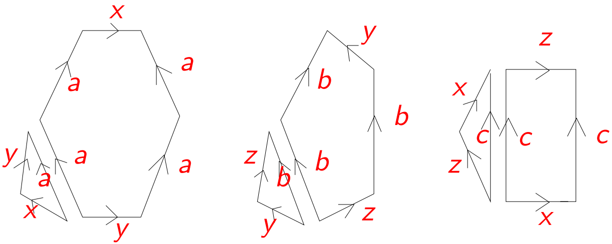
```
gap> F:=FreeGroup(6);;x:=F.1;;y:=F.2;;z:=F.3;; a:=F.4;;
b:=F.5;;c:=F.6;;
gap> rels:=[a^-1*x*y, b^-1*y*z, c^-1*z*x,
a^2*x*(y*a^2)^-1, b^2*(z*b*y)^-1, c*z*(x*c)^-1];;

gap> IsAspherical(F,rels);
true
```

Analysis of GAP command



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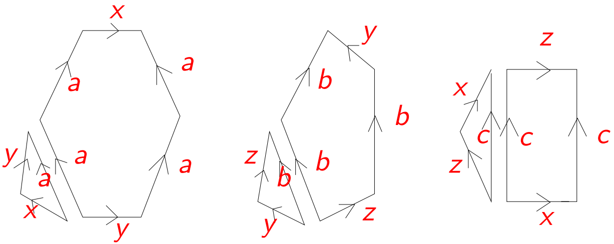

Can we give each of the cells of X a euclidean metric such that any loop in $\tilde{X}$ has at least 2π radians? If yes, then X is aspherical.

This can be phrased as a linear programming problem which is tackled using POLYMAKE software.

Example 2

The Artin group $G =< x, y, z : xyx = yxy, xz = zx,$
$yzyzy = zyzyz >$ has

$$H^n(G, \mathbb{Z}) = \mathbb{Z}\ (0 \leq n \leq 3), H^n(G, \mathbb{Z}) = 0\ (n \geq 4).$$

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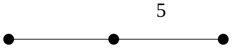
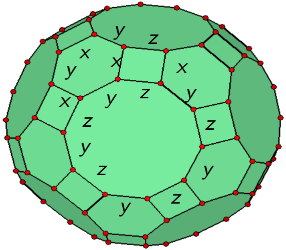
```
gap> D:=[[1,[2,3]],[2,[3,5]]];;
gap> GroupCohomology(D,1);
[ 0 ]
gap> GroupCohomology(D,2);
[ 0 ]
gap> GroupCohomology(D,3);
[ 0 ]
```

Analysis of GAP command

Let X be the "canonical" quotient of the 3-dimensional polytope





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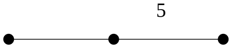

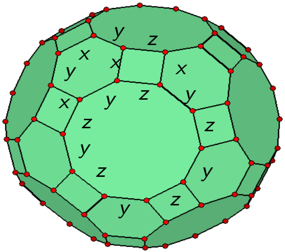

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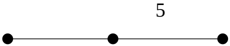

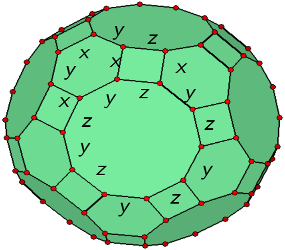

This polytope is $P(W_G)$ where $W_G = G/ < x^2 = y^2 = z^2 = 1 >$.

It was shown independently by C. Squier and M. Salvetti that such a space X is aspherical. Hence $H^*(G, \mathbb{Z}) = H^*(X, \mathbb{Z})$.

Salvetti's proof.

W_G acts "by reflections" on a real vector space V and on an open *Tits' cone* $I \subset V$.

$$\mathcal{A} = \text{set of reflecting hyperplanes},$$

$$M(W_G) = I \oplus \mathbf{i}V \setminus (\cup_{H \in \mathcal{A}} H \oplus \mathbf{i}H) \subset \mathbb{C} \otimes V = V \oplus \mathbf{i}V,$$

$$N(W_G) = M(W_G)/W_G.$$

$K(\pi, 1)$-Conjecture

$N(W_G)$ is an Eilenberg-Mac Lane space $K(G, 1)$.

Deligne proved this for finite W_G.

Salvetti proved $X \simeq N(W_G)$ (for any finite W_G).

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Salvetti proved $X \simeq N(W_G)$ (for any finite W_G).

R. Charney and M. Davis proved $X \simeq N(W_G)$ for arbitrary Coxeter groups W_G.

Conjecture

The Artin group $G' =< w, x, y, z : wxw = xwx, wy = yw, wzw = zwz, xyx = yxy, xz = zx,\ yzyzy = zyzyz >$ has

$$H^1(G', \mathbb{Z}) = H^2(G', \mathbb{Z}) = \mathbb{Z},$$

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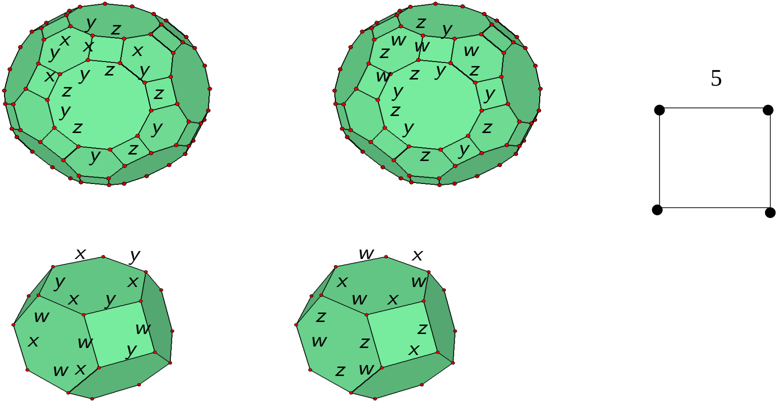
Computer evidence

```
gap> D:=[[1,[2,3],[4,3]],[2,[3,3]],[3,[4,5]]];;
gap> GroupCohomology(D,3);
[ 2, 2, 0, 0 ]

#   etc.
```

Analysis of computer evidence

Let X' be the "canonical" path-connected quotient of the four polytopes:



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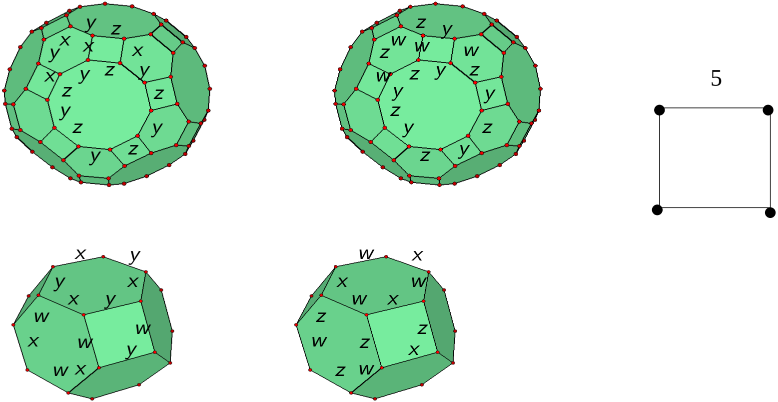

It is not known if X' is aspherical.

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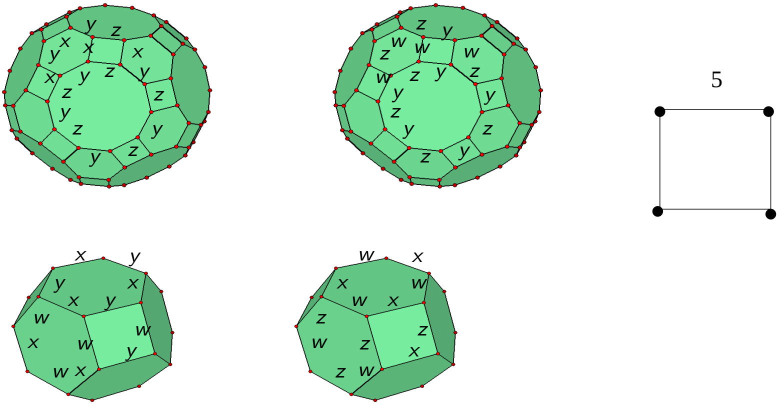

It is not known if X' is aspherical. Remark: $W_{G'}$ is infinite whereas W_G was finite.

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The Coxeter group W_G is obtained by adding the relation $x^2 = 1$ for each generator.

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An Artin group admits a finite dimensional classifying space whose k-cells correspond to those subsets of the generators of W_G that generate finite subgroups.

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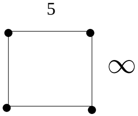
Theorem (E & Skoldberg)
Let Γ be the graph of G. If the conjecture holds for every full subgraph of Γ involving no ∞-edges, then it holds for Γ.

Illustration

The conjecture holds for

$$G =< w, x, y, z : wxw = xwx, wy = yw, xyx = yxy,$$
$$xz = zx, yzyzy = zyzyz >.$$



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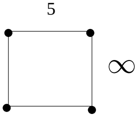
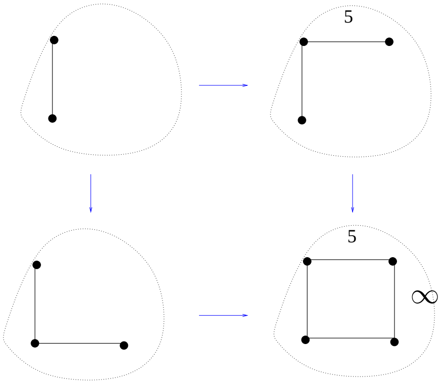
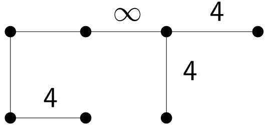


Proof:



Example 3

The Artin group G defined by the Coxeter graph



has integral cohomology groups

$$
\begin{array}{lll}
H^0(G,\mathbb{Z}) \cong \mathbb{Z}, & H^1(G,\mathbb{Z}) \cong \mathbb{Z}^5, & H^2(G,\mathbb{Z}) \cong \mathbb{Z}^{11}, \\
H^3(G,\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^{14}, & H^4(G,\mathbb{Z}) \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}^{12}, & H^5(G,\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^6, \\
H^6(G,\mathbb{Z}) \cong \mathbb{Z}, & H^n(G,\mathbb{Z}) = 0 \ (n \geq 7). &
\end{array}
$$

THE END

THANK YOU!