Computing cohomology of groups

In honour of Jacques Calmet
Logrono, Februrary 2008
and
In honour of Hans-Joachim Baues
Bonn, March 2008

Graham Ellis
NUI Galway, Ireland

(Supported by Marie Curie MTKD-CT-2006-042685)

Problem

Compute the (co)homology

$$H^*(G, A) = H^*(BG, A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

of a discrete group G.

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Compute the (co)homology

$$H^*(G, A) = H^*(BG, A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

of a discrete group G.

More generally, G could be a homotopy 2-type (crossed module).

E X A M P L E S 1

(Number Crunching)

Theorem

The Mathieu group M_{23} has trivial integral homology $H_n(M_{23}, \mathbb{Z}) = 0$ in dimensions $n = 1, 2, 3$.

Proof.

R.J. Milgram, "The cohomology of the Mathieu group M_{23}", *J. Group Theory* 3 (2000), no. 1, 7–26.

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Computer Proof.

```
gap> GroupHomology(MathieuGroup(23),1);
[  ]
gap> GroupHomology(MathieuGroup(23),2);
[  ]
gap> GroupHomology(MathieuGroup(23),3);
[  ]
```

Analysis of computer proof

- $|M_{23}| = 10200960 = 2^7.3^2.5.7.23$

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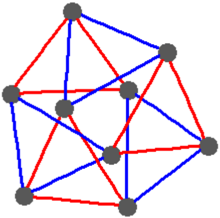
- $X^1_{(3)} =$ 

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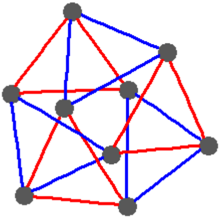
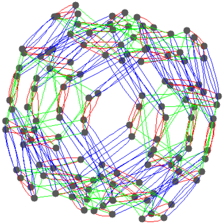
- $X^1_{(3)} =$  $X^1_{(2)} =$ 

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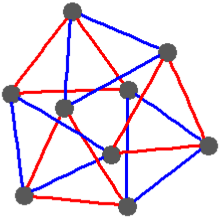
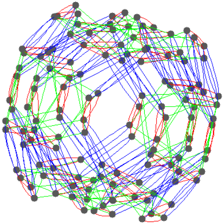
- $X^1_{(3)} =$  $X^1_{(2)} =$ 
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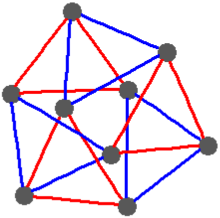
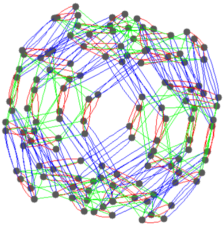
- $X^1_{(3)} =$  $X^1_{(2)} =$ 
- $C_*(X_{(p)})$ is a free $\mathbb{Z}P$-resolution of $\mathbb{Z}$.
- During the construction of $X_{(p)}$ record an explicit contracting homotopy $h_* \colon C_*(X_{(p)}) \to C_{*+1}(X_{(p)})$.

- There is a surjection $H_n(P,\mathbb{Z}) \to H_n(G,\mathbb{Z})_{(p)}$ whose kernel is described (Cartan-Eilenberg) in terms of induced homomorphisms

$$\iota_x \colon H_n(P,\mathbb{Z}) \to H_n(xPx^{-1},\mathbb{Z})$$

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- ι_x constructed using h_*.

Conjecture

Any classifying space for an n generator Coxeter group G, whose 2-skeleton corresponds to the standard Coxeter presentation of G, must have at least $\frac{(n+k-1)!}{(n-1)!k!}$ k-dimensional cells.

[M. Salvetti, "Cohomology of Coxeter groups", Topology and its Applications 118 (2002), 199-208]

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Disproof

```
gap> F:=FreeGroup(3);;x:=F.1;;y:=F.2;;z:=F.3;;
gap> S_4:=F/[x^2, y^2, z^2, (x*z)^2, (y*z)^3, (x*y)^3];;
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gap> R:=ResolutionSmallFpGroup(S_4,3);;
gap> Dimension(R)(3);
7
```

Conjecture

Any classifying space for an n generator Coxeter group G, whose 2-skeleton corresponds to the standard Coxeter presentation of G, must have at least $\frac{(n+k-1)!}{(n-1)!k!}$ *k-dimensional cells.*

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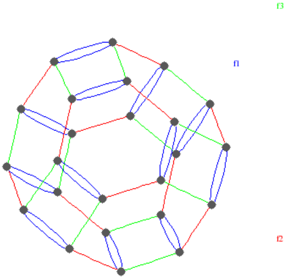
The 3-cells in R are a subset of those in Salvetti's complex. For example:

```
gap> IdentityAmongRelationsDisplay(R,7);
```



E X A M P L E 2

(Twisted Tensor Product)

Theorem

For an odd prime p the group $K_p = \ker(\mathrm{SL}_2(\mathbb{Z}_{p^3}) \to \mathrm{SL}_2(\mathbb{Z}_p))$ has third integral homology group of exponent p^3.

Proof.

W. Browder and J. Pakianathan, "Cohomology of uniformly powerful p-groups", *Trans. Amer. Math. Soc.* 352 (2000), no. 6, 2659–2688. $\square$

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Computer Proof.

```
gap> K5:=MaximalSubgroups(SylowSubgroup(
                              SL(2,Integers mod 5^3),5))[2];;
gap> GroupHomology(K5,3);
[ 5, 5, 5, 5, 5, 5, 125 ]
```

Analysis of computer proof

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$$1 \to N \to G \to Q \to 1$$

  and

  - a free $\mathbb{Z}N$-resolution $R_*^N \to \mathbb{Z}$
  - a free $\mathbb{Z}Q$-resolution $R_*^Q \to \mathbb{Z}$

  then the differential on the tensor product of chain complexes $R^N \otimes_{\mathbb{Z}} R^Q$ can be perturbed to produce a free $\mathbb{Z}G$-resolution

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- There are several explanations of this perturbation. We use a Lemma of CTC Wall .

Let A be a ring. (e.g. $A = \mathbb{Z}G$.) Let

$$C_*\colon \ \to C_n \to C_{n-1} \to \cdots \to C_0$$

be an A-resolution of some A-module M, where the A-modules C_n are **not** assumed to be free.

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Suppose that, for each p, we have a **free** A-resolution of C_p

$$D_{p*}: \ \to D_{p,q} \to D_{p,q-1} \to \cdots \to D_{p,0} \to C_p$$

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Lemma (C.T.C. Wall)

There exists a free A-resolution $R_* \to M$ with

$$R_n = \bigoplus_{p+q=n} D_{p,q}.$$

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
 & D_{32} \arrow[d] & D_{22} \arrow[d, "d^0"] & D_{12} \arrow[d] & D_{02} \arrow[d] \\
 & D_{31} \arrow[d] & D_{21} \arrow[d, "d^0"] & D_{11} \arrow[d] & D_{01} \arrow[d] \\
 & D_{30} \arrow[d] & D_{20} \arrow[d] & D_{10} \arrow[d] & D_{00} \arrow[d] \\
 {} \arrow[r] & C_3 \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r] & C_0
\end{tikzcd}
$$

$$d^0 d^0 = 0$$

$$
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 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
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but for $d^1 d^1 \neq 0$

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but for $d^2 d^2 \neq 0$ etc

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There is a free A-resolution $R_* \to M$ with

$$R_n = \bigoplus_{p+q=n} D_{p,q}$$

and boundary homomorphism

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A contracting homotopy on R_* can be constructed using homotopies on D_{p*} and C_*

E X A M P L E 3

(Linear Algebra & Gröbner Bases)

Theorem

The mod 2 cohomology $H^n(M_{11}, \mathbb{Z}_2)$ of the Mathieu group M_{11} is a vector space of dimension equal to the coefficients of x^n in the Poincaré series

$$(x^4 - x^3 + x^2 - x + 1)/(x^6 - x^5 + x^4 - 2x^3 + x^2 - x + 1)$$

for all n.

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P.J. Webb, "A local method in group cohomology" *Comment. Math. Helv.* 62 (1987), no. 1, 135–167. □

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Computer proof for $n \leq 20$.

```
gap> PoincareSeriesPrimePart(MathieuGroup(11),2,20);
(x^4-x^3+x^2-x+1)/(x^6-x^5+x^4-2*x^3+x^2-x+1)
```

Analysis of computer proof

- For the field $\mathbb{F}$ of p elements any free $\mathbb{F}G$-module $(\mathbb{F}G)^n$ can be treated as a vector space of dimension $n \times |G|$. Linear algebra can be used to determine minimal generators for kernels of $\mathbb{F}G$-homomorphisms.

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But how do we compute a Poincaré series which is known to be correct in all degrees?

Theorem (Well-Known)

The quaternion group G of order 8 has cohomology ring

$$H^*(G, \mathbb{F}) = \mathbb{F}[x, y, e] / < x^2 + xy + y^2,\ y^3 >$$

where $\mathbb{F}$ is the field of two elements, x, y have degree 1 and e has degree 2.

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Computer Proof. (Paul Smith)

- The central extension $1 \to C_2 \to G \to C_2 \times C_2 \to 1$ yields the LHS spectral sequence

$$E_2^* = H^*(C_2 \times C_2, \mathbb{F}) \otimes H^*(C_2, \mathbb{F}) = \mathbb{F}[x,y,z] \Longrightarrow H^*(G,\mathbb{F})$$

Theorem (Well-Known)

The quaternion group G of order 8 has cohomology ring

$$H^*(G,\mathbb{F}) = \mathbb{F}[x,y,e]/ < x^2+xy+y^2,\ y^3 >$$

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- Our CTC Wall resolution defines the derivation $d_2\colon E_2^* \to E_2^*$ by $d_2(x) = d_2(y) = 0,\ d_2(z) = x^2+xy+y^2$.

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$$E_4^* = E_\infty^* = \mathbb{F}[x, y, z^2]/ < x^2 + xy + y^2,\ y^3 >$$

E X A M P L E S 5

(Convex Hulls & Perturbations)

Theorem (Dutour & E)

$$H_3(M_{24}, \mathbb{Z}) \cong \mathbb{Z}_{12}$$

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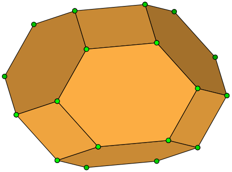
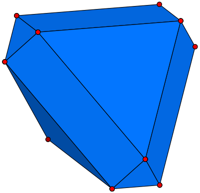
- $M_{24} < S_{24}$ acts on $\mathbb{R}^{24}$ by permuting the standard basis. Let $v = (1, 2, 3, 4, 5, 0, \ldots, 0) \in \mathbb{R}^{24}$ and compute the polytope

$$P_v(M_{24}) = \mathrm{ConvexHull}(v^{M_{24}}).$$

- Low dimensional illustrations using POLYMAKE: $P_v(S_4)$ and $P_v(A_4)$ with $v = (1, 2, 3, 4)$





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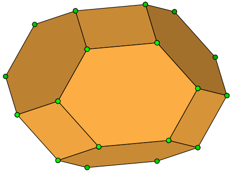 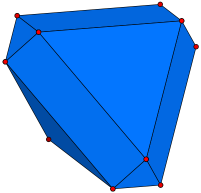

- The computation of $P = P_v(M_{24})$ is helped by using the 5-transitivity of M_{24} to first prove that P is simple. For $v = (1, 2, 3, 4, 5, 0, \ldots, 0)$ we have

$$P_v(M_{24}) = P_v(S_{24}).$$

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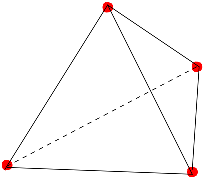

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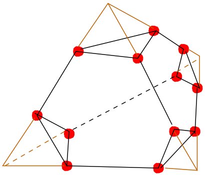

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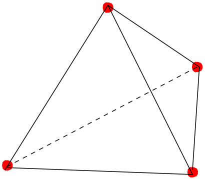

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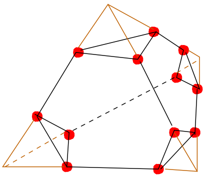

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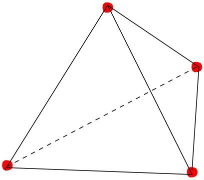

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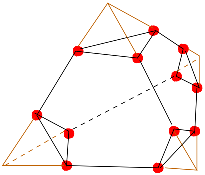

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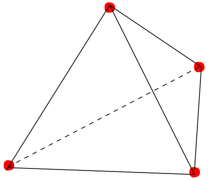

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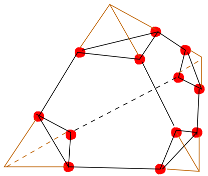

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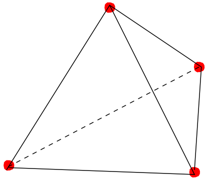

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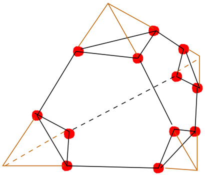

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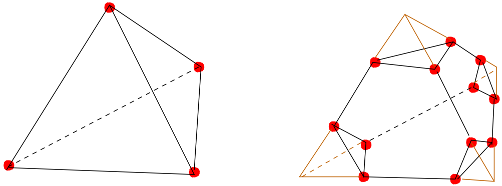

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- Copies of $C_*(P)$ can be spliced together to form a periodic $\mathbb{Z}M_{24}$-resolution of $\mathbb{Z}$, but this is not free. For instance $|Stab(v)| = 48$.
- We use CTC Wall's lemma to enlarge this resolution into a free one.

Remark

For a finite reflection group W generated by simple reflections $x_1, \cdots, x_n$ the polytope $P_v(W)$ has

$$k\text{-faces} \leftrightarrow \text{cosets of subgroups} < x_{i_1}, \cdots, x_{i_k} >$$

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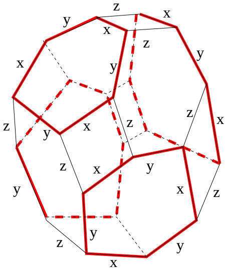

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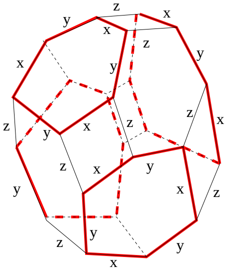

So CTC Wall's lemma imples a free $\mathbb{Z}W$-resolution with $\frac{(n+k-1)!}{(n-1)!k!}$ k-dimensional cells.

EXAMPLES 6

(Convex Hulls for some infinite groups)

Similar computational techniques can be applied to crystallographic groups:

Theorem

The ring $H^*(M, \mathbb{Z})$ has now been computed for each of the 62 7-dimensional Hantzsche-Wendt manifolds M.

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Computer proof. (Marc Roeder)

- By definition M is a flat manifold with point group $(C_2)^6$.
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- Let $G = \pi_1 M$. Choose $v \in \mathbb{R}^n$ and use POLYMAKE to determine a fundamental domain

$$D(G, v) = \{x \in \mathbb{R}^n : ||x - v|| < ||x - g(v)|| \text{ for all } g \in G\}.$$

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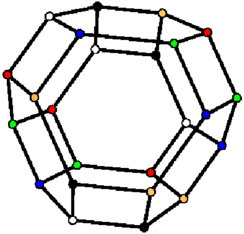
- The resulting $C_*(\mathbb{R}^n)$ is a free $\mathbb{Z}G$-resolution.
- A (partial) contracting homotopy can be constructed on $C_*(\mathbb{R}^n)$ and used for the cup product.

Low dimensional illustrations.

G=SpaceGroup(3,9) has fundamental domain

$D(G,v) =$ 

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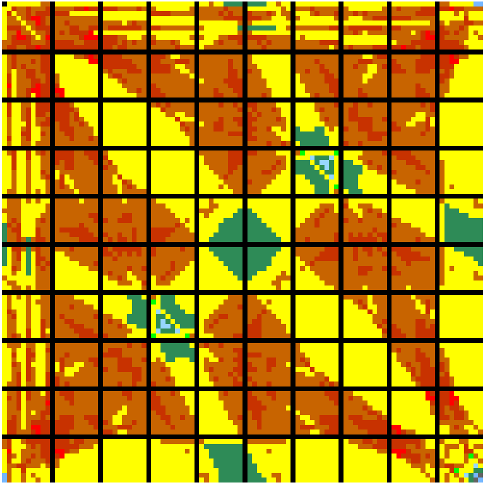
The domains depend only on the x and y coordinates of v.



EXAMPLES 7

(Asphericity testing)

Example (Well-known)

The presentation of the affine braid group

$$G = \langle x, y, z : xyx = yxy, yzy = zyz, xzx = zxz \rangle$$

corresponds to an aspherical 2-dimensional CW-space X.

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Computer proof

First change the CW-decomposition on X by considering

$$G = \langle x, y, z, a, b, c, : a = xy, b = yz, c = zx, ax = ya, by = zb, cz = xc \rangle.$$

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```
gap> F:=FreeGroup(6);;x:=F.1;;y:=F.2;;z:=F.3;; a:=F.4;;
b:=F.5;;c:=F.6;;
gap> rels:=[a^-1*x*y, b^-1*y*z, c^-1*z*x, a*x*(y*a)^-1,
b*y*(z*b)^-1, c*z*(x*c)^-1];;

gap> IsAspherical(F,rels);
true
```

Analysis of proof

Can we give each of the cells of X a euclidean metric such that any loop in $\tilde{X}$ has at least 2π radians? If yes, then curvature/euler characteristic arguments would imply that X is non-positively curved and hence aspherical.

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This can be phrased as a linear programming problem which is tackled using POLYMAKE software.

E X A M P L E S 8

(Polytopal Combinatorics)

Theorem

The Artin group $G =< x, y, z : xyx = yxy, xz = zx,$ $yzyzy = zyzyz >$ *has*

$$H^n(G, \mathbb{Z}) = \mathbb{Z} (0 \leq n \leq 3), H^n(G, \mathbb{Z}) = 0 \ (n \geq 4).$$

Proof.

C. Landi, "Cohomology rings of Artin groups", Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 11 no. 1 (2000), 41-65. □

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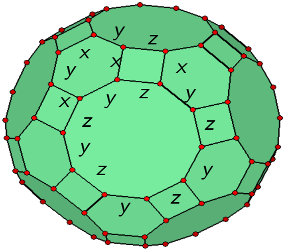
Computer proof

```
gap> D:=[[1,[2,3]],[2,[3,5]]];;
gap> GroupCohomology(D,1);
[ 0 ]
gap> GroupCohomology(D,2);
[ 0 ]
gap> GroupCohomology(D,3);
[ 0 ]
```

Analysis of computer proof

Let X be the "canonical" quotient of the 3-dimensional polytope



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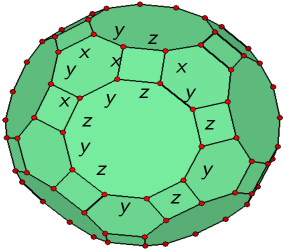

This polytope is $P(W_G)$ where $W_G = G / < x^2 = y^2 = z^2 = 1 >$.

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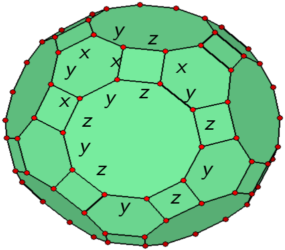

This polytope is $P(W_G)$ where $W_G = G / < x^2 = y^2 = z^2 = 1 >$.

It was shown independently by C. Squier and M. Salvetti that such a space X is aspherical. Hence $H^*(G, \mathbb{Z}) = H^*(X, \mathbb{Z})$.

Conjecture

The Artin group $G' =< w, x, y, z : wxw = xwx, wy = yw, wzw = zwz, xyx = yxy, xz = zx,\ yzyzy = zyzyz >$ has

$$H^1(G',\mathbb{Z}) = H^2(G',\mathbb{Z}) = \mathbb{Z},$$

$$H^3(G',\mathbb{Z}) = (\mathbb{Z}_2)^2 \oplus \mathbb{Z}^2,\ \ H^n(G',\mathbb{Z}) = 0\ (n \geq 4).$$

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Computer evidence

```
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gap> GroupCohomology(D,1);
[ 0 ]
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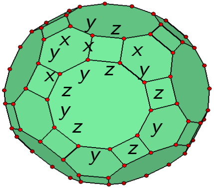
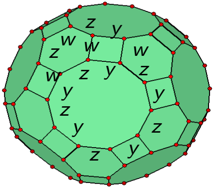
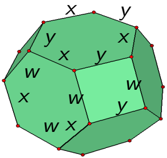
```

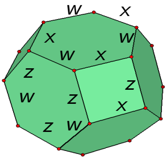
Analysis of computer evidence

Let X' be the "canonical" path-connected quotient of the four polytopes:



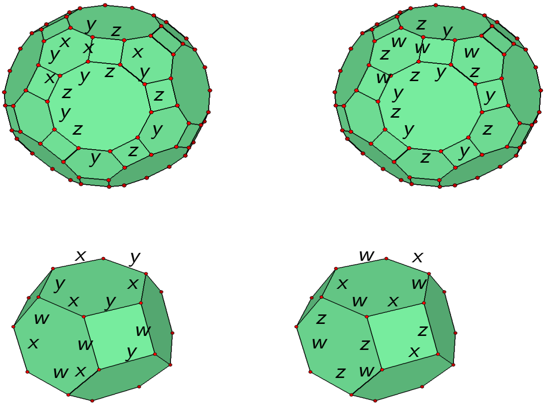






Analysis of computer evidence

Let X' be the "canonical" path-connected quotient of the four polytopes:



It is not known if X' is aspherical.

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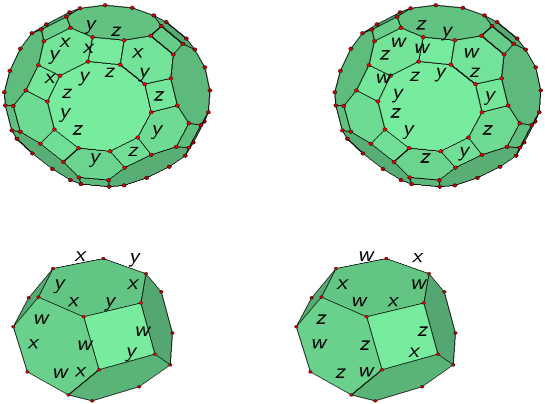

It is not known if X' is aspherical. Remark: $W_{G'}$ is infinite whereas W_G was finite.

HOMOTOPY 2-TYPES

(Ideas, No Examples Yet!)

A *homotopy n-type* is represented by a connected CW-space X with $\pi_i X = 0$ for $i \geq n+1$.

$$B\colon \text{(groups)} \xrightarrow{\simeq} \text{(homotopy } 1-\text{types)}$$

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Whitehead, Baues, Loday et al.:

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Problem. (with Ana Romero)

Compute $H^*(G, A) = H^*(BG, A)$.

$$
\begin{tikzcd}
B\colon (\mathrm{cat}^1-\mathrm{groups}) \arrow[r, "\mathcal{N}"] & (\mathrm{simplicial\ groups}) \arrow[d, "\mathcal{N}"] & \\
& (\mathrm{bisimplicial\ sets}) \arrow[r, "\Delta"] & (\mathrm{simplicial\ sets})
\end{tikzcd}
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$H_*(G,\mathbb{Z})$ is the homology of the total complex of the bicomplex:

$$
\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & F\mathcal{N}_2\mathcal{N}_2(G) \arrow[r] \arrow[d] & F\mathcal{N}_2\mathcal{N}_1(G) \arrow[r] \arrow[d] & F\mathcal{N}_2\mathcal{N}_0(G) \arrow[d] \\
{} \arrow[r] & F\mathcal{N}_1\mathcal{N}_2(G) \arrow[r] \arrow[d] & F\mathcal{N}_1\mathcal{N}_1(G) \arrow[r] \arrow[d] & F\mathcal{N}_1\mathcal{N}_0(G) \arrow[d] \\
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We could replace each column by

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where $R_*^{\mathcal{N}_j(G)}$ is an arbitrary free $\mathbb{Z}\mathcal{N}_j(G)$-resolution of $\mathbb{Z}$.

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Also, for which classes of groups do there exist small functorial free resolutions?

```
gap> F:=FreeGroup(4);; G:=NilpotentQuotient(F,2);;
gap> GroupHomology(G,8);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0 ]
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Conjecture

The integral homology of the free nilpotent group G of class c is equal to the integral homology of the associated Lie algebra LG.

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Holds for $c = 2$. [Yu. V. Kuz'min & Yu. S. Semenov, Mat. Sb., 1998]

Theorem

The free nilpotent group G of class two on 4 generators has integral cohomology groups

$$\begin{array}{lll} H^1(G,\mathbb{Z}) \cong \mathbb{Z}^4, & H^2(G,\mathbb{Z}) \cong \mathbb{Z}^{20}, & H^3(G,\mathbb{Z}) \cong \mathbb{Z}^{56}, \\ H^4(G,\mathbb{Z}) \cong \mathbb{Z}^{84}, & H^5(G,\mathbb{Z}) \cong \mathbb{Z}_3^4 \oplus \mathbb{Z}^{90}, & H^6(G,\mathbb{Z}) \cong \mathbb{Z}_3^4 \oplus \mathbb{Z}^{84}, \\ H^7(G,\mathbb{Z}) \cong \mathbb{Z}^{56}, & H^8(G,\mathbb{Z}) \cong \mathbb{Z}^{20}, & H^9(G,\mathbb{Z}) \cong \mathbb{Z}^4, \\ H^{10}(G,\mathbb{Z}) \cong \mathbb{Z}, & H^n(G,\mathbb{Z}) = 0 \ (n \geq 11). & \end{array}$$

The ring $H^*(G,\mathbb{Z})$ is minimally generated by: 4 classes in degree 1, 20 classes in degree 2, 36 classes in degree 3 and 20 classes in degree 4.

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Conjecture:

In the cohomology of the free nilpotent group of class 2 on ≥ 4 generators the first torsion occurs in dimension 5 and is 3-torsion.

THE END

THANK YOU!