Computing cohomology rings of p-groups: using spectral sequences for completion tests

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Recipe for calculating mod-p group cohomology rings

For a p-group G where k is the field of p elements, there is a mechanistic method to compute the group cohomology:

1. Calculate n terms of a minimal projective resolution of the trivial module k
$$P_n \to \ldots \to P_2 \to P_1 \to kG \to k \to 0$$
where $P_i = (kG)^{b_i}$ and b_i are the *Betti numbers*.
2. Use the resolution to find the generators of the cohomology ring (those of degree $\leq n$)
3. Compute the relations among these generators

Implementations:

- Carlson (Magma)
- Green/King (SAGE)
- Bishop (GAP - Crime)
- Ellis/Smith (GAP - HAP /HAPprime)

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Existing approach: Completion test (Carlson/Benson)

- Set of theorems on the properties of mod-p cohomology rings
- Compute cohomology ring for chosen n, then apply tests
- Repeat with larger n if necessary
- Implementations: Carlson (Magma), Green/King (SAGE)
- But tests are complicated – no GAP implementation yet

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We propose an alternative method which can determine a minimal 'large enough n'.

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mod-p group cohomology rings are finitely-generated

How are we sure that there exists a ‘large-enough n’?

- mod-p group cohomology rings are finitely-generated
- Proof: (Len Evens (1961)) The Lyndon-Hochschild-Serre spectral sequence converges in a finite number of steps

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- i.e. construct the Lyndon-Hochschild-Serre spectral sequence

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Talk outline

1. Spectral sequences
2. Computing Lyndon-Hochschild-Serre spectral sequence
3. Working with derivations in mod p
4. Using the L-H-S spectral sequence to compute cohomology ring
5. Example computations

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Definitions: bigraded Abelian group

Definition

A *bigraded Abelian group* is a family of Abelian groups $E = \{E_{p,q}\}_{p,q \in \mathbb{Z}}$

$$
\begin{array}{ccccc|l}
 & & & & & q \\
E_{4,3} & E_{3,3} & E_{2,3} & E_{1,3} & E_{0,3} & \\
E_{4,3} & E_{3,2} & E_{2,2} & E_{1,2} & E_{0,2} & \\
E_{4,3} & E_{3,1} & E_{2,1} & E_{1,1} & E_{0,1} & \\
E_{4,3} & E_{3,0} & E_{2,0} & E_{1,0} & E_{0,0} & \\
\hline
p & & & & &
\end{array}
$$

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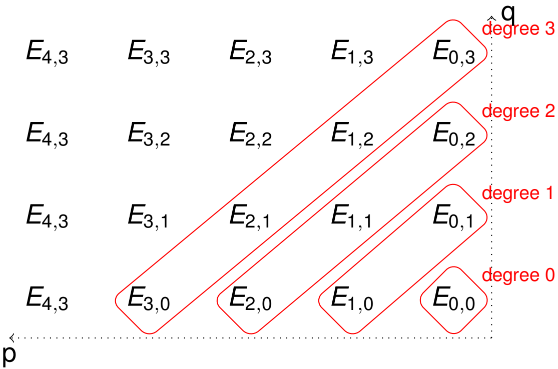

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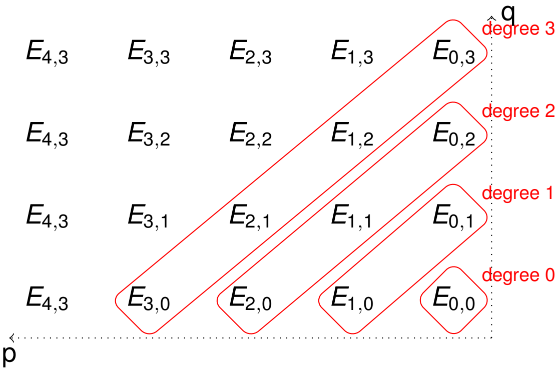

Typically, the index $n = p + q$ is called the degree.

A spectral sequence is a sequence E^r of bigraded Abelian groups. We call each group E^r a *sheet*.

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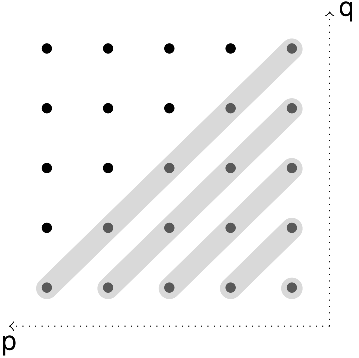
Definitions: differential

Definition

A *differential* $d : E \to E$ is a family of morphisms of groups, parameterised by r, such that $d \circ d = 0$ and

$$d^r_{p,q} : E_{p,q} \to E_{p-r,q+r-1}$$



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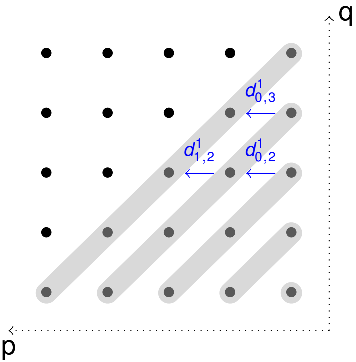

$r = 1 \qquad d^1_{p,q} : E_{p,q} \longrightarrow E_{p-1,q}$

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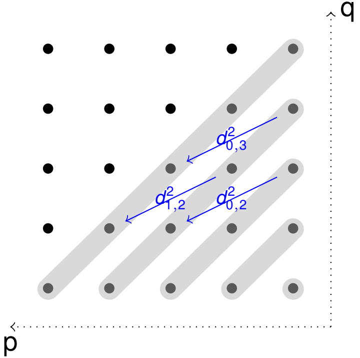

$r = 2 \qquad d^2_{p,q} : E_{p,q} \longrightarrow E_{p-2,q+1}$

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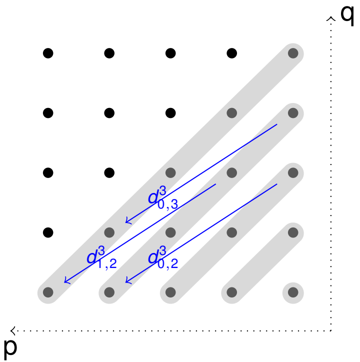

$r = 3 \qquad d^3_{p,q} : E_{p,q} \to E_{p-3,q+2}$

Definitions: homology of E

Recall that the differentials $d : E \to E$ square to zero:

$$d_{p-r,r+r-1} \circ d_{p,q} = 0$$

which implies that

$$\operatorname{im} d_{p-r,r+r-1} \subseteq \ker d_{p,q}$$

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Definition

The *homology* of E is the bigraded group $H(E) = H(E,d) = \{H_{p,q}(E)\}_{p,q\in\mathbb{Z}}$ with

$$H_{p,q}(E) = \frac{\ker d_{p,q}}{\operatorname{im} d_{p-r,r+r-1}}$$

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Definitions: spectral sequence

Definition

A *spectral sequence* $E = (E^r, d^r)$ is a collection of three sequences. For each index $r \geq 1$, we have

1. a sheet (bigraded Abelian group) $E^r = \{E^r_{p,q}\}_{p,q \in \mathbb{Z}}$
2. a set of differentials on the sheet $d^r = \{d^r_{p,q}\}_{p,q \in \mathbb{Z}}$
3. the isomorphism $E^{r+1} \cong H(E^r, d^r)$

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- Given an initial sheet E^1 and the differentials d^r for $r \geq 1$, we can generate all other sheets
- Each sheet is contained within the previous one, i.e. $E^{r+1}_{p,1} \subseteq E^r_{p,q}$
- In some useful cases the spectral sequence converges to the *final groups* $E^\infty_{p,q} = \bigcap_{r\geq 1} E^r_{p,q}$

The Lyndon-Hochschild-Serre spectral sequence

Let $N \rightarrowtail G \twoheadrightarrow Q$ be a central extension of the group G. Then:

- the sheet $E^2_{p,q} = H^p(Q, H^q(N, k))$
- the sheets represent a graded algebra, not just a module
- under multiplication the differentials are *derivations*, i.e. $d(xy) = d(x)y + xd(y)$
- additively, the spectral sequence converges to the group cohomology ring: $E^\infty_{p,q} \cong_k H^{p+q}(G, k)$

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Issues:

1. We don't get the true cohomology ring, only something of the same 'size'
2. To calculate this spectral sequence, we need to be able to find and work with derivations

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The L-H-S spectral sequence as a completion criterion

- The final ring E_*^∞ is not necessarily isomorphic to $H^*(G, k)$
- But E_*^∞ *is* an associate graded ring of $H^*(G, k)$
- The generators and relations for E_*^∞ occur in the same degrees as for $H^*(G, k)$

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Completion criterion

- The maximum generator or relation degree in E_*^∞ is the same as the maximum generator or relation degree in $H^*(G, k)$.
- This is the same as the minimum ‘large enough’ n

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Approach: Use the L-H-S spectral sequence to find n, then calculate $H^*(G,k)$ in the traditional manner. We will then know that we have all the generators and relations.

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By contrast, the Carlson/Benson completion criteria only give a sufficient n, not the smallest value

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The challenge of derivations

Finding and working with the differentials is usually the challenge when working with spectral sequences.

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Problem 1: Finding the differentials

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Problem 1: Finding the differentials

In the Lyndon-Hochschild-Serre spectral sequence, the differentials can be found from the resolutions for Q and N using a theorem of C.T.C. Wall

- Already (mostly) implemented in the HAP package for GAP (Ellis)

The challenge of derivations

Problem 2: Finding the kernel of a differential

In calculating $H(E) = \ker d / \operatorname{im} d$, the hard part is finding the kernel of the differential.
Recall that here, the differentials are derivations:

$$
\begin{aligned}
d(x + y) &= d(x) + d(y) \\
d(xy) &= d(x)y + xd(y)
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Thus:

- A derivation is a module homomorphism
- A derivation is *not* a ring homomorphism
- The kernel of a derivation is not an ideal
- Computing $\ker(d)$ is not easy in the general case
- Kernel of derivation is not a standard part of computational algebra packages (e.g. Singular , CoCoA)

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The kernel of a derivation in characteristic 2

Consider a polynomial ring $R = \mathbb{F}_2[x, y]$ and a derivation $d : R \to R$. Then,

$$d(r^2) = rd(r) + d(r)r = 2rd(r) = 0 \mod 2 \qquad \forall r \in R$$

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Consider $R \supseteq S = \mathbb{F}_2[x^2, y^2]$. Then R is a finitely-generated, free S-module:

$$R = S \oplus S.x \oplus S.y \oplus S.xy$$

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So d is an S-module homomorphism.

The argument can be made in any positive characteristic p, using $S = \mathbb{F}_p[x^p, y^p]$.

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Computing the kernel of a derivation in characteristic 2

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1. Write d as an S-module homomorphism d_S:
   - Let S be the subring $\mathbb{F}_2[x_1^2, \dots x_n^2]$
   - Write the images of the module generators in S-module form
   - Convert relations into a module ideal

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   - Gives a (redundant) generating set for $\ker d$

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   - Gives a (redundant) generating set for $\ker d$
4. Find a minimal set of generators and relations for $\ker d$
   - The generating set can be large, but we have some methods to deal with this

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Recipe for calculating mod-p group cohomology ring

To compute the mod-p group cohomology ring for a group G

1. Compute the Lyndon-Hochschild-Serre spectral sequence to find E_*^∞
   1.1 For a central extension $N \rightarrowtail G \twoheadrightarrow Q$, compute resolutions and cohomology rings for N and Q (note that $|N| < |G|$ and $|Q| < |G|$)
   1.2 Construct E_*^2 sheet ($E_*^2 = H^*(Q, \mathbb{F}) \otimes H^*(N, \mathbb{F})$)
   1.3 Calculate derivations d^2 (calculated using C.T.C. Wall)
   1.4 Repeat
      - Let $E_*^{r+1} = H(E^r, d^r)$ (calculated using kernel of derivation)
      - Calculate next derivations d^r

   Until all future derivations $d^i = 0 \quad \forall \quad i \geq r$
2. Let n be the maximum degree in generators and relations of E_*^∞
3. Compute the cohomology ring for G
   3.1 Compute minimal resolution of length n for G
   3.2 Compute cohomology ring from resolution in the normal manner

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Example: Group cohomology ring for Q_8

The central extension for Q_8 is $C_2 \rightarrowtail Q_8 \twoheadrightarrow C_2 \times C_2$

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The central extension for Q_8 is $C_2 \rightarrowtail Q_8 \twoheadrightarrow C_2 \times C_2$

1. Calculate minimal projective resolutions for $N = C_2$ and $Q = C_2 \times C_2$

$$\mathbb{F}N \to \mathbb{F}N \to \mathbb{F}N \to \mathbb{F}N \to \mathbb{F} \to 0$$

$$\mathbb{F}Q^4 \to \mathbb{F}Q^3 \to \mathbb{F}Q^2 \to \mathbb{F}Q \to \mathbb{F} \to 0$$

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1. Calculate minimal projective resolutions for $N = C_2$ and $Q = C_2 \times C_2$

$$\mathbb{F}N \to \mathbb{F}N \to \mathbb{F}N \to \mathbb{F}N \to \mathbb{F} \to 0$$
$$\mathbb{F}Q^4 \to \mathbb{F}Q^3 \to \mathbb{F}Q^2 \to \mathbb{F}Q \to \mathbb{F} \to 0$$

2. Calculate group cohomology ring presentations for N and Q

$$H^*(C_2, \mathbb{F}) = \mathbb{F}[e]$$
$$H^*(C_2 \times C_2, \mathbb{F}) = \mathbb{F}[x, y]$$

Example: Group cohomology ring for Q_8

3. Create E^2_* sheet: $E^2_* = \mathbb{F}[x, y] \otimes \mathbb{F}[e]$

E^2 sheet:

$\vdots$

e^3

e^2

e

1

graded basis of $H^*(N, \mathbb{F}) = \mathbb{F}[e]$

graded basis of $H^*(Q, \mathbb{F}) = \mathbb{F}[x, y]$

$\ldots \quad x^3, x^2y, xy^2, y^3 \quad x^2, xy, y^2 \quad x, y \quad 1$

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Example: Group cohomology ring for Q_8

3. Create E^2_* sheet: $E^2_* = \mathbb{F}[x, y] \otimes \mathbb{F}[e]$

E^2 sheet:

$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	$\vdots$
$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	e^3
$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	e^2
$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	e
$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	1
$\ldots$	x^3, x^2y, xy^2, y^3	x^2, xy, y^2	x, y	1	

graded basis of $H^*(Q, \mathbb{F}) = \mathbb{F}[x, y]$

graded basis of $H^*(N, \mathbb{F}) = \mathbb{F}[e]$

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Example: Group cohomology ring for Q_8

3. Create E^2_* sheet: $E^2_* = \mathbb{F}[x, y] \otimes \mathbb{F}[e]$
4. Calculate derivations d^2 for ring generators

E^2 sheet:

$$\begin{tikzcd}
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & \vdots \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & e^3 \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & e^2 \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} \arrow[dll] & e \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & 1 \\
\dots & x^3, x^2y, xy^2, y^3 & x^2, xy, y^2 & x, y & 1 &
\end{tikzcd}$$

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Example: Group cohomology ring for Q_8

3. Create E^2_* sheet: $E^2_* = \mathbb{F}[x,y] \otimes \mathbb{F}[e]$
4. Calculate derivations d^2 for ring generators

E^2 sheet:

$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	$\vdots$
$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	e^3
$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	e^2
$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	e
$\mathbb{F}^5$	$\mathbb{F}^4$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	1
$\dots$	x^3, x^2y, xy^2, y^3	x^2, xy, y^2	x, y	1	

$x^2 + xy + y^2$

$$d^2(x) = 0$$
$$d^2(y) = 0$$
$$d^2(e) = x^2 + xy + y^2$$

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Example: Group cohomology ring for Q_8

5. Compute homology to create E^3 sheet: $E^3_* = H(E^2, d^2)$

E^2 sheet:

$$
\begin{array}{cccccc}
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & \vdots \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & e^3 \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & e^2 \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & e \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & 1 \\
\ldots & x^3, x^2y, xy^2, y^3 & x^2, xy, y^2 & x, y & 1 &
\end{array}
$$

$$d^2(x) = 0$$
$$d^2(y) = 0$$
$$d^2(e) = x^2 + xy + y^2$$

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Example: Group cohomology ring for Q_8

5. Compute homology to create E^3 sheet: $E^3_* = H(E^2, d^2)$

E^2 sheet:

$$
\begin{array}{cccccc}
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & \vdots \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & e^3 \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & e^2 \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & e \\
\mathbb{F}^5 & \mathbb{F}^4 & \mathbb{F}^3 & \mathbb{F}^2 & \mathbb{F} & 1 \\
\ldots & x^3, x^2y, xy^2, y^3 & x^2, xy, y^2 & x, y & 1 &
\end{array}
$$

$$d^2(x) = 0$$
$$d^2(y) = 0$$
$$d^2(e) = x^2 + xy + y^2$$

$$H(E^2, d^2) = E^3 = \frac{\mathbb{F}[x, y, e^2]}{x^2 + xy + y^2}$$

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Example: Group cohomology ring for Q_8

5. Compute homology to create E^3 sheet: $E^3_* = H(E^2, d^2)$

E^3 sheet:

$$
\begin{array}{cccccc}
\mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F} & \vdots \\
0 & 0 & 0 & 0 & 0 & e^3 \\
\mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F} & e^2 \\
0 & 0 & 0 & 0 & 0 & e \\
\mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F} & 1 \\
\dots & xy^2, y^3 & xy, y^2 & x, y & 1 &
\end{array}
$$

$$H(E^2, d^2) = E^3 = \frac{\mathbb{F}[x, y, e^2]}{x^2 + xy + y^2}$$

Example: Group cohomology ring for Q_8

5. Compute homology to create E^3 sheet: $E^3_* = H(E^2, d^2)$
6. Calculate derivations d^3 for ring generators

E^3 sheet:

$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	$\vdots$
0	0	0	0	0	e^3
$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	e^2
0	0	0	0	0	e
$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	1
$\ldots$	xy^2, y^3	xy, y^2	x, y	1	

y^3

$$H(E^2, d^2) = E^3 = \frac{\mathbb{F}[x, y, e^2]}{x^2 + xy + y^2}$$

$$d^3(x) = 0$$
$$d^3(y) = 0$$
$$d^3(e) = y^3$$

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Example: Group cohomology ring for Q_8

7. Compute homology to create E^4 sheet: $E^4_* = H(E^3, d^3)$

E^3 sheet:

$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	$\vdots$
0	0	0	0	0	e^3
$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	e^2
0	0	0	0	0	e
$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	1
$\dots$	xy^2, y^3	xy, y^2	x, y	1	

$$d^3(x) = 0$$
$$d^3(y) = 0$$
$$d^3(e) = y^3$$

$$H(E^3, d^3) = E^4 = \frac{\mathbb{F}[x, y, e^4]}{x^2 + xy + y^2, \quad y^3}$$

Example: Group cohomology ring for Q_8

7. Compute homology to create E^4 sheet: $E^4_* = H(E^3, d^3)$

E^4 sheet:

0	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	e^4
0	0	0	0	0	e^3
0	0	0	0	0	e^2
0	0	0	0	0	e
0	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	1
	xy^2, y^3	xy, y^2	x, y	1	

$$H(E^3, d^3) = E^4 = \frac{\mathbb{F}[x, y, e^4]}{x^2 + xy + y^2, \quad y^3}$$

Example: Group cohomology ring for Q_8

7. Compute homology to create E^4 sheet: $E^4_* = H(E^3, d^3)$
8. Calculate derivations d^4 for ring generators

E^4 sheet:

$$
\begin{array}{cccccc}
0 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F} & e^4 \\
0 & 0 & 0 & 0 & 0 & e^3 \\
0 & 0 & 0 & 0 & 0 & e^2 \\
0 & 0 & 0 & 0 & 0 & e \\
0 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F} & 1 \\
 & xy^2,\ y^3 & xy,\ y^2 & x,\ y & 1 &
\end{array}
$$

$$H(E^3, d^3) = E^4 = \frac{\mathbb{F}[x, y, e^4]}{x^2 + xy + y^2,\quad y^3}$$

$$d^4(x) = 0$$
$$d^4(y) = 0$$
$$d^4(e) = 0$$

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Example: Group cohomology ring for Q_8

Now it is easy:

- $E_*^4 = \dfrac{\mathbb{F}[x,y,e^4]}{x^2+xy+y^2, \ \ y^3}$
- $d^4(.) = 0$
- Hence $E_*^5 = H(E^4, d^4) = E_*^4$

0	0	0	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	e^4
0	0	0	0	0	0	0	e^3
0	0	0	0	0	0	0	e^2
0	0	0	0	0	0	0	e
0	0	0	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	1
			xy^2, y^3	xy, y^2	x, y	1	

Example: Group cohomology ring for Q_8

Now it is easy:

- $E^4_* = \frac{\mathbb{F}[x,y,e^4]}{x^2+xy+y^2,\ \ y^3}$
- $d^4(.) = 0$
- Hence $E^5_* = H(E^4, d^4) = E^4_*$

$$
\begin{array}{cccccccc}
0 & 0 & 0 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F} & e^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\
0 & 0 & 0 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F}^2 & \mathbb{F} & 1 \\
 & & & xy^2,\ y^3 & xy,\ y^2 & x,\ y & 1 &
\end{array}
$$

And every higher differential is also zero, so

$$E^4_* = E^5_* = \ldots = E^\infty_*$$

and we have convergence:

$$E^\infty_* = \frac{\mathbb{F}[x,y,e^4]}{x^2+xy+y^2,\ \ y^3}$$

Example: Group cohomology ring for Q_8

$$E_*^\infty = \frac{\mathbb{F}[x, y, e^4]}{x^2 + xy + y^2, \quad y^3} \quad \cong_{\mathbb{F}} H^*(G, \mathbb{F})$$

This is only *additively* the same as the cohomology. Even so, we can already state:

- The Hilbert-Poincaré series is $(t^2 + t + 1)/(-t^3 + t^2 - t + 1)$

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Example: Group cohomology ring for Q_8

$$E_*^\infty = \frac{\mathbb{F}[x, y, e^4]}{x^2 + xy + y^2, \quad y^3} \quad \cong_{\mathbb{F}} H^*(G, \mathbb{F})$$

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- The Hilbert-Poincaré series is $(t^2 + t + 1)/(-t^3 + t^2 - t + 1)$

But this is enough to find a value for n, the resolution length needed to calculate the true cohomology ring

- The generators are of degrees 1, 1 and 4
- The relations are of degrees 2 and 3

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Example: Group cohomology ring for Q_8

$$E_*^\infty = \frac{\mathbb{F}[x, y, e^4]}{x^2 + xy + y^2, \quad y^3} \quad \cong_{\mathbb{F}} H^*(G, \mathbb{F})$$

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But this is enough to find a value for n, the resolution length needed to calculate the true cohomology ring

- The generators are of degrees 1, 1 and 4
- The relations are of degrees 2 and 3
- To capture all generators and relations, need $n \geq 4$

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Example: Group cohomology ring for Q_8

$$E_*^\infty = \frac{\mathbb{F}[x, y, e^4]}{x^2 + xy + y^2, \quad y^3} \quad \cong_{\mathbb{F}} H^*(G, \mathbb{F})$$

This is only *additively* the same as the cohomology. Even so, we can already state:

- The Hilbert-Poincaré series is $(t^2 + t + 1)/(-t^3 + t^2 - t + 1)$

But this is enough to find a value for n, the resolution length needed to calculate the true cohomology ring

- The generators are of degrees 1, 1 and 4
- The relations are of degrees 2 and 3
- To capture all generators and relations, need $n \geq 4$
- Calculate cohomology ring using resolution of length 4
- In fact, in this case $E_*^\infty = H^*(G, \mathbb{F})$

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Example: GAP computation for a group of order 128

```
gap> G := SmallGroup(128, 928);;
gap> A := ModPCohomologyRingPresentationSpectralSequence(G); StringTime(time);
<graded algebra>
" 0:00:54.575"
gap> n := MaximumNeededDegreeInPresentation(A);
6
gap> ModPCohomologyRingPresentation(G, n);StringTime(time);
[ GF(2)[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9],
  [ x_6*x_8, x_5*x_7, x_5*x_6, x_3*x_5, x_2*x_7+x_3*x_8, x_2*x_6, x_1*x_8,
      x_1*x_7, x_1*x_4, x_1*x_3, x_1*x_2,
      x_3^2*x_9+x_3*x_4*x_7+x_4^2*x_6+x_7^2, x_2*x_3*x_9+x_3*x_4*x_8+x_7*x_8,
      x_2^2*x_9+x_2*x_4*x_8+x_4^2*x_5+x_8^2 ], [ 1, 1, 1, 2, 2, 2, 3, 3, 4 ] ]
" 0:00:30.434"
```

For this arbitrary group of order 128:

- Spectral sequence converges after 6 sheets
- 54 seconds to calculate spectral sequence
- 30 seconds to calculate cohomology ring with $n = 6$
- Memory requirement: $<$ 100Mb
- $E_*^\infty \ncong H^*(G, k)$

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Example: GAP computation for a group of order 128

```
gap> G := SmallGroup(128, 928);;
gap> A := ModPCohomologyRingPresentationSpectralSequence(G); StringTime(time);
<graded algebra>
" 0:00:54.575"
gap> n := MaximumNeededDegreeInPresentation(A);
6
gap> ModPCohomologyRingPresentation(G, n);StringTime(time);
[ GF(2)[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9],
  [ x_6*x_8, x_5*x_7, x_5*x_6, x_3*x_5, x_2*x_7+x_3*x_8, x_2*x_6, x_1*x_8,
      x_1*x_7, x_1*x_4, x_1*x_3, x_1*x_2,
      x_3^2*x_9+x_3*x_4*x_7+x_4^2*x_6+x_7^2, x_2*x_3*x_9+x_3*x_4*x_8+x_7*x_8,
      x_2^2*x_9+x_2*x_4*x_8+x_4^2*x_5+x_8^2 ], [ 1, 1, 1, 2, 2, 2, 3, 3, 4 ] ]
" 0:00:30.434"
```

For this arbitrary group of order 128:

- Spectral sequence converges after 6 sheets
- 54 seconds to calculate spectral sequence
- 30 seconds to calculate cohomology ring with $n = 6$
- Memory requirement: $<$ 100Mb
- $E_*^\infty \not\cong H^*(G, k)$
- Carlson convergence criteria here needs $n = 12$
- Calculating ring using $n = 12$ takes $>$ 12 hours and $>$ 1.6Gb

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Example GAP timings for all groups of order 32

```
gap> L := List(AllSmallGroups(32), x->ModPCohomologyRingPresentationSpectralSequence(x));;
gap> StringTime(time);
"0:36:24.408"
gap> degs := List(L,
> HAPPRIME_MaximumNeededDegreeInPresentation);
[ 4, 2, 2, 4, 6, 8, 12, 4, 6, 6, 2, 2, 2, 6, 2, 6, 2, 6, 4, 2, 4, 2, 6, 2,
  4, 4, 2, 6, 6, 6, 6, 6, 2, 4, 2, 6, 4, 2, 6, 4, 4, 6, 10, 2, 2, 4, 4, 4, 8
 ]
gap> List([1..51], i->ModPCohomologyRingPresentation(SmallGroup(32, i), degs[i]));
gap> StringTime(time);
" 0:00:29.838"
gap>
gap> List(AllSmallGroups(32), x->ModPCohomologyRingPresentation(x, 12));;
StringTime(time);
"0:03:06.116"
```

Using spectral sequences:

- 34 minutes to calculate all spectral sequences
- Largest number of sheets is 10; median is 6
- Largest n is 12; median n is 4
- 30 seconds to calculate all cohomology rings

Using Carlson completion criteria:

- Largest n is 12; median n is 6
- 7 minutes to calculate all cohomology rings to necessary length
- Time does not include completion test

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Conclusion

- Evens' finite presentation proof can be used constructively
- Gives a completion criteria for mod-p cohomology ring calculations

- The Lyndon-Hochschild-Serre spectral sequence can be efficiently computed
- Derivations over mod-p rings can be treated as module homomorphisms
- The spectral sequence gives the additive structure for $H^*(G, k)$, and a minimal 'large enough' value for n
- Have an efficient implemention in the GAP package HAPprime

Thanks

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Example: GAP output for cohomology computation

```
gap> G := SmallGroup(64, 31);;
gap> A := ModPCohomologyRingPresentationSpectralSequence(G);
#I  E_2 = GF(2)[ z, y, x ]/[ z^2+z*y ] x GF(2)[ w, v ]/[ w^2 ]
#I    with generator degrees [ 1, 1, 2 ] and [ 1, 2 ] respectively
#I    d_2(z) = zero
#I    d_2(y) = zero
#I    d_2(x) = zero
#I    d_2(w) = z^2 = z*y mod I
#I    d_2(v) = 0*Z(2)
#I  E_3 = GF(2)[ v, x, y, z, z*w+y*w ]/[ z^2*w^2+y^2*w^2, z^2*w+z*y*w, z^2, z*\
y ]
#I    d_3(v) = z^3+y*x = y*x mod I
#I    d_3(x) = zero
#I    d_3(y) = zero
#I    d_3(z) = zero
#I    d_3(z*w+y*w) = zero
#I  E_4 = GF(2)[ z*w+y*w, z, y, x, z*v, v^2 ]/[ z^2*v^2, z*y*v, y*x, z^2*v, z*\
y, z^2, z^2*w*v+z*y*w*v, z^2*w+z*y*w, z^2*w^2+y^2*w^2 ]
#I    d_4(z*w+y*w) = zero
#I    d_4(z) = zero
#I    d_4(y) = zero
#I    d_4(x) = zero
#I    d_4(z*v) = zero
#I    d_4(v^2) = 0*Z(2)
```

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Example: GAP output for cohomology computation

```
#I  E_5 = GF(2)[ z*w+y*w, z, y, x, z*v, v^2 ]/[ z^2*v^2, z*y*v, y*x, z^2*v, z*\
y, z^2, z^2*w*v+z*y*w*v, z^2*w+z*y*w, z^2*w^2+y^2*w^2 ]
#I    d_5(z*w+y*w) = zero
#I    d_5(z) = zero
#I    d_5(y) = zero
#I    d_5(x) = zero
#I    d_5(z*v) = zero
#I    d_5(v^2) = z^5+z^3*x+y^3*x+y*x^2 = 0*Z(2) mod I
#I  E_6 = GF(2)[ z*w+y*w, z, y, x, z*v, v^2 ]/[ z^2*v^2, z*y*v, y*x, z^2*v, z*\
y, z^2, z^2*w*v+z*y*w*v, z^2*w+z*y*w, z^2*w^2+y^2*w^2 ]
#I    d_6(z*w+y*w) = zero
#I    d_6(z) = zero
#I    d_6(y) = zero
#I    d_6(x) = zero
#I    d_6(z*v) = zero
#I    d_6(v^2) = zero
#I  E_inf = GF(2)[ z*w+y*w, z, y, x, z*v, v^2 ]/[ z^2*v^2, z*y*v, y*x, z^2*v, \
z*y, z^2, z^2*w*v+z*y*w*v, z^2*w+z*y*w, z^2*w^2+y^2*w^2 ]
#I  Renaming indeterminates and sorting into increasing degree
[[ GF(2)[z,y,x,w,v,u],
  [ v^2, x*v, x^2, y*v, y*w, z*v, z*x, z*y, z^2 ], [ 1, 1, 2, 2, 3, 4 ] ]
```

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Example: GAP output for cohomology computation

So, $E_*^\infty = \dfrac{\mathbb{F}_2[z,y,x,w,v,u]}{(v^2,\ xv,\ x^2,\ yv,\ yw,\ zv,\ zx,\ zy,\ z^2)}$

```
gap> List(A[2], x->DegreeOfPolynomial(A, x));
[ 6, 5, 4, 4, 3, 4, 3, 2, 2 ]
```

The maximum degree in the presentation is $n = 6$.

```
gap> B := ModPCohomologyRingPresentation(G, 6);
[ GF(2)[z,y,x,w,v,u],
  [ w^2, y*v, y*x, z*v+x*w, z*w, z*y, z^2, x^3+v^2, z*x^2+w*v ],
  [ 1, 1, 2, 2, 3, 4 ] ]
gap>
gap> RingPresentationsAreIsomorphic(A, B);
false
```

$$H^*(G, \mathbb{F}_2) = \frac{\mathbb{F}_2[z,y,x,w,v,u]}{(w^2,\ yv,\ yx,\ zv+xw,\ zw,\ zy,\ z^2,\ x^3+v^2,\ zx^2+wv)}$$

and in this case $E_*^\infty \not\cong H^*(G, \mathbb{F})$.

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