# Computational Homology IV

Berlin, August 2010

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Topologists study

$$[X, Y] := \{f\colon X \to Y\}/\text{homotopy}$$

using functors

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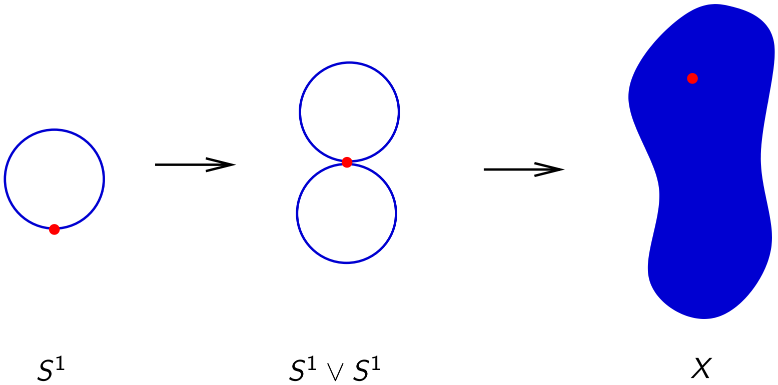
Assumptions for the lecture:
$X, Y$ connected CW-spaces with chosen base-point; maps preserve base-point.

The set $\pi_1(X) := [S^1, X]$ has a group structure:



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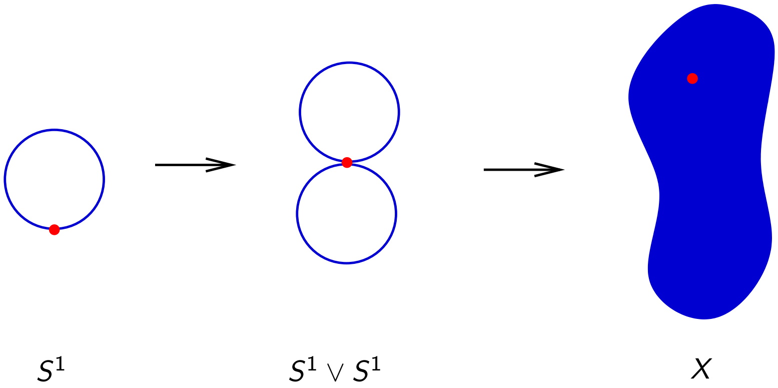
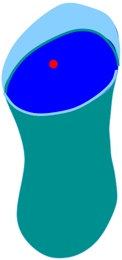



Theorem: If $X$ is 1-dimensional then $[X, Y] \cong [\pi_1 X, \pi_1 Y]$.

Seifert-van Kampen: $\pi_1$: *Top* $\rightarrow$ *Groups* preserves certain colimits.

For instance, if $X = V \cup W$

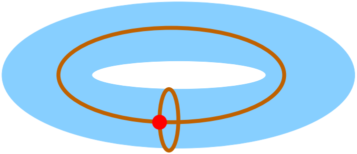


$X$

with $V, W$ $V \cap W$ path-connected then:

$$\pi_1(V \cup W) = \pi_1(V) *_{\pi_1(V \cap W)} \pi_1(W)$$

# Computing $\pi_1(S^1 \times S^1)$



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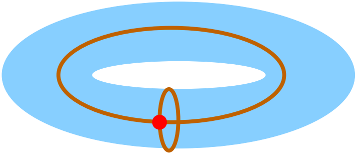

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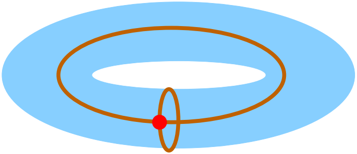

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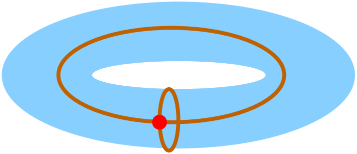

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$\pi_1((S^1 \vee S^1) \cup e^2) = \pi_1(S^1 \vee S^1) *_{\pi_1 S^1} \pi_1(*) = \langle x, y \mid xyx^{-1}y^{-1} = 1 \rangle$ (van Kampen)

$$\pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$$

Whitehead: If $X$ is 2-dimensional then $[X, Y] \cong [C_*(\tilde{X}), C_*(\tilde{Y})]$.

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What functor

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could satisfy

$$[X, Y] \cong [\Pi(X), \Pi(Y)]$$

when $\dim(X) = 3$?

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Can it compute homotopy groups

$$\pi_n(X) = [S^n, X] \qquad (n = 1, 2, 3)?$$

A map $f\colon Y \to X$ is connected if $X, Y$ and

$$Z = \text{Homotopy fibre}(f)$$

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A crossed module is a group homomorphism $\partial\colon M \to G$ with action $(g, m) \mapsto {}^g m$ satisfying

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4. Any map of spaces $f\colon Y \to X$ yields a crossed module $\pi(f)\colon \pi_1(Z) \to \pi_1(Y)$ where $Z$ is the homotopy fibre of $f$.

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Whitehead: If $X$ is 2-dimensional then

$$[X, Y] \cong [\pi(X^1 \to X), \pi(Y^1 \to Y)]$$

Theorem (Brown & Higgins): The functor

$$\begin{array}{cccc} \pi & : & (\textit{topological maps}) & \longrightarrow & (\textit{crossed modules}) \\ & & Y \to X & \mapsto & \pi_1 Z \to \pi_1 Y \end{array}$$

preserves certain colimits.

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For instance, pushouts of connected maps

$$\begin{array}{ccc} (U \to U') & \longrightarrow & (W \to W') \\ \downarrow & & \downarrow \\ (V \to V') & \longrightarrow & (X \to X') \end{array}$$

yield pushouts of crossed modules

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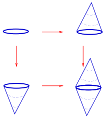
Example: The pushout

$$\begin{array}{ccc} (S^1 \to S^1) & \longrightarrow & (S^1 \to E^2) \\ \downarrow & & \downarrow \\ (S^1 \to E^2) & \longrightarrow & (S^1 \to S^2) \end{array}$$



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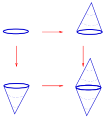

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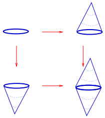

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$$\pi_2(S^2) \cong \ker(\partial) \cong \mathbb{Z}$$

A map of maps (or commutative square)

$$\begin{array}{ccc} U & \longrightarrow & W \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

is connected if the spaces and homotopy fibres

$$\begin{array}{ccccc} Z_X & \longrightarrow & Z_{UV} & \longrightarrow & Z_{WX} \\ \downarrow & & \downarrow & & \downarrow \\ Z_{UW} & \longrightarrow & U & \longrightarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ Z_{VX} & \longrightarrow & V & \longrightarrow & X \end{array}$$

are path connected.

For connected

$$\begin{array}{ccc} U & \longrightarrow & W \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

with $U, V, W$ classifying spaces, the group homomorphisms

$$\begin{array}{ccc} \pi_1(Z_X) & \xrightarrow{\partial} & \pi_1(Z_{UV}) \\ \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial'} \\ \pi_1(Z_{UW}) & \xrightarrow{\partial} & \pi_1(U) \end{array}$$

determine $\pi_n(X)$ for $n = 1, 2, 3$.

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Baues: $[X, Y] \cong [\pi(X), \pi(Y)]$ when $\dim(X) \leq 3$.

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with $U = B(G), V = B(G/N), W = B(G/M)$ yields a pushout

$$\pi \begin{pmatrix} U & W \\ V & X \end{pmatrix} = \begin{array}{ccc} M \otimes N & \rightarrow & N \\ \downarrow & & \downarrow \\ M & \rightarrow & G \end{array}$$

Thus the pushout

$$\begin{array}{ccc} B(G) & \longrightarrow & B(G/M) \\ \downarrow & & \downarrow \\ B(G/N) & \longrightarrow & X \end{array}$$

has

$$\begin{aligned} \pi_1(X) &= G/MN \\ \pi_2(X) &= (M \cap N)/[M, N] \\ \pi_3(X) &= \ker(M \otimes N \to G) \end{aligned}$$

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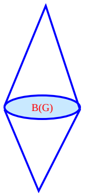

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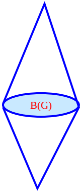


$$\pi_3(S^2) = \ker(\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{0} \mathbb{Z})$$

Dennis, Brown, Loday:

The group $M \otimes N$ is generated by symbols

$$m \otimes n$$

for $m \in M, n \in N$, subject to relations

$$mm' \otimes n = ({}^{m}m' \otimes {}^{m}n)\,(m \otimes n)$$

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and $\partial$ is a crossed module.

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- $G$ perfect implies $H_2(G, \mathbb{Z}) \cong \ker(G \otimes G \to G)$.

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Machine computations of $M \otimes N$ (mainly $M = N$):

Brown, Johnson & Robertson (1987)
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Example: $\pi_3(\, SB(\, GL_4(\mathbb{Z}_3)\,)\,) = \mathbb{Z}_2$

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gap> ThirdHomotopyGroupOfSuspensionB(GL(4,3));
[ 2 ]
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Folklore:

$$\begin{array}{ccc} \text{category objects in} & \simeq & \text{crossed modules} \\ \text{category of groups} & & \end{array}$$

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| | | |
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| category objects in category of groups | $\simeq$ | crossed modules |

$$s, t \colon C \longrightarrow Ob(C)$$

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category objects in category of groups $\simeq$ crossed modules

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$$C \times_{Ob(C)} C \xrightarrow{\circ} C \qquad\qquad \ker(s) \xrightarrow{t} Ob(C)$$

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| $s, t\colon C \longrightarrow Ob(C)$ | | |
| $C \times_{Ob(C)} C \xrightarrow{\circ} C$ | | $\ker(s) \xrightarrow{t} Ob(C)$ |

A fibration $f: X \longrightarrow Y$ yields the category object:

$$s, t\colon \pi_1(X \times_Y X) \longrightarrow \pi_1(X)$$

$$a \circ b := a\, s(a^{-1})\, b$$

Definition/Theorem (Loday):

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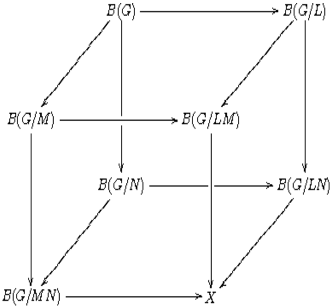
Definition/Theorem (E & Steiner)

$$cat^n - groups \quad \simeq \quad crossed\ \text{n}-cubes$$

Applications (E & Mikhailov). A colimit of classifying spaces:

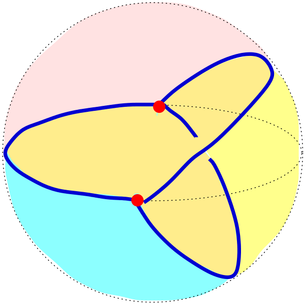



$$\pi_1(X) \cong G/LMN$$

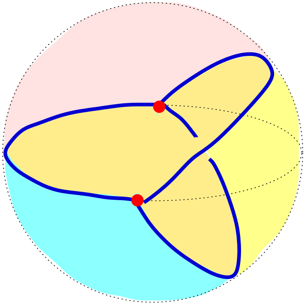
$$\pi_2(X) \cong LM \cap MN \,/\, M(L \cap N)$$

$$\pi_3(X) \cong L \cap M \cap N \,/\, [L, M \cap N][M, L \cap N][N, L \cap M]$$

$$\pi_4(X) \cong \ker(\partial \colon \otimes(L, M, N) \to G)$$



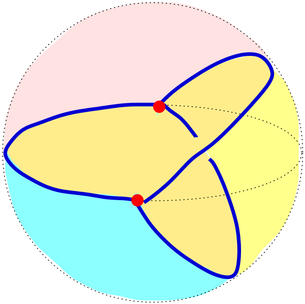
$$S^2(X) = CS(X) \cup CS(X) \cup CS(X)$$



$S^2(X) =$
$CS(X) \cup CS(X) \cup CS(X)$

$X = B(G)$ yields

$$\pi_4(S^2(B(G))) \cong \ker(\partial\colon \otimes(G, G, G) \to G)$$



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Lemma: $\otimes(G, G, G) \cong G \otimes G / \{(g \otimes g')(g' \otimes g) = 1\}$

$$G := \langle x, y \mid \emptyset \rangle$$

$$L = \langle x \rangle^G, \quad M = \langle y \rangle^G, \quad N = \langle xy \rangle^G$$

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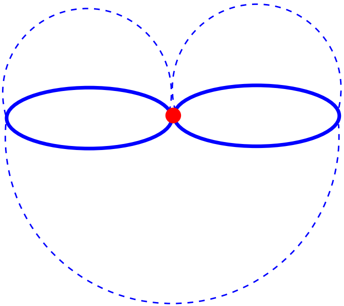

$$B(G/LM) \cup B(G/LN) \cup B(G/LN)$$
$$= S^2$$

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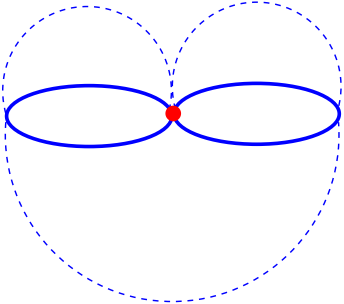

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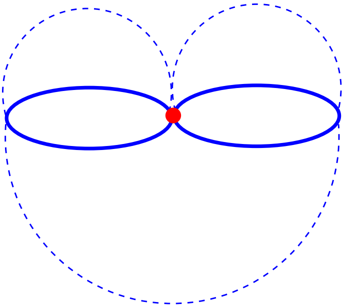

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Theorem (Wu): Analogous formula for $\pi_{n+1}(S^2)$ from

$$\langle x_1, \ldots, x_n \mid x_1 = x_2 = \ldots x_n = x_1 x_2 \cdots x_n = 1 \rangle$$