Computational Homology III

Berlin, August 2010

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NUI Galway, Ireland

A homotopy equivalence data

$$(L,d) \overset{p}{\underset{i}{\leftrightarrows}} (M,d),\ \ h \qquad (*)$$

consists of chain complexes L, M, quasi-isomorphisms i, p and a homotopy $ip - 1 = dh + hd$. A perturbation on $(*)$ is a homomorphism $\epsilon \colon M \to M$ of degree -1 such that $(d+\epsilon)^2 = 0$.

PERTURBATION LEMMA: If $A = (1-\epsilon h)^{-1}\epsilon$ exists then

$$(L,d') \overset{p'}{\underset{i'}{\leftrightarrows}} (M,d+\epsilon),\ \ h' \qquad (**)$$

is a homotopy equivalence data where

$$i' = i + hAi, \quad p' = p + pAh, \quad h' = h + hAh, \quad d' = d + pAi \quad .$$

M. Crainic, "On the perturbation lemma, and deformations", 2004

Recall (yet again!)

G discrete group

X contractible space on which G acts fixed point freely.

$$H_*(G, \mathbb{Z}) = H_*(X/G, \mathbb{Z})$$

Example: $G = \langle x, y | yxy = x \rangle$

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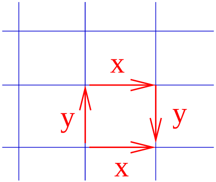

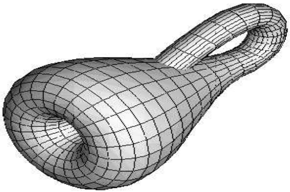

$X = \mathbb{R}^2$

X/G = Klein bottle

Cell structure: $X/G = e^0 \cup e^1_x \cup e^1_y \cup e^2$

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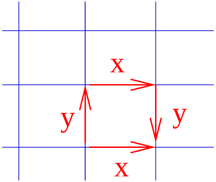

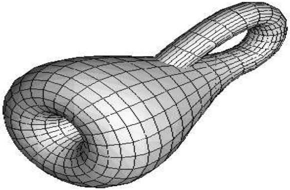

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$$H_n(G, \mathbb{Z}) = H_n(X/G, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}, & n = 1 \\ 0, & n \geq 2 \end{cases}$$

A *Bieberbach group* G is a discrete group of isometries of $\mathbb{R}^n$ for which $\mathbb{R}^n/G$ is a compact manifold.

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Its cohomology can be computed from any fundamental domain

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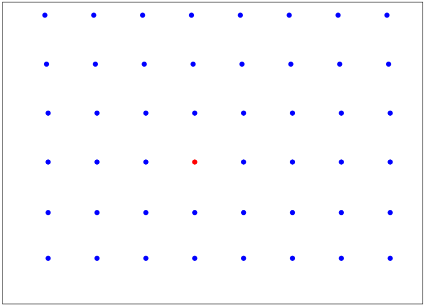

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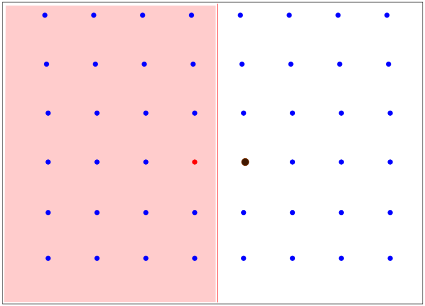

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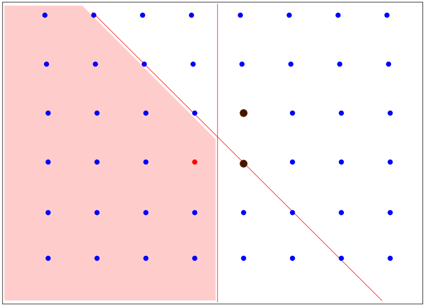

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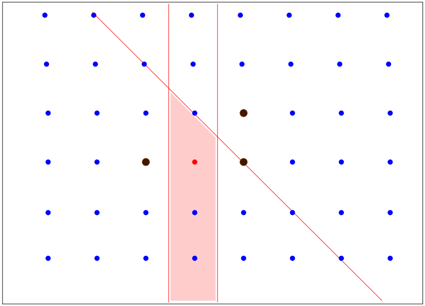

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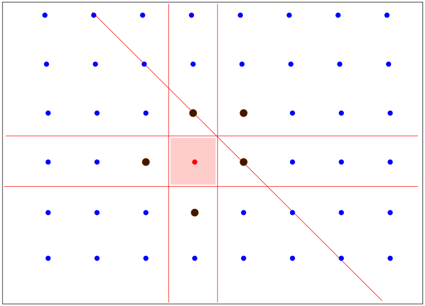

The embedding

$$\begin{array}{ccc} \mathbb{R}^2 & \hookrightarrow & \mathbb{R}^3 \\ (x, y) & \mapsto & (x, y, 1) \end{array}$$

is used to represent elements of $G = \langle x, y | yxy = x \rangle$ as matrices

$$\begin{pmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

GAP Example

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G:=SpaceGroup(3,165);;
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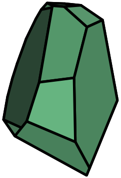

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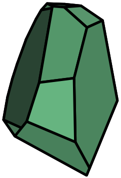

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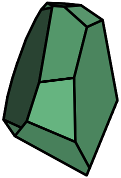

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gap> R:=ResolutionBieberbachGroup(G);;

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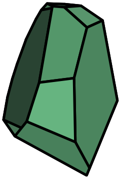

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gap> C:=TensorWithIntegers(R);;

gap> Homology(C,3);
[ 0 ]
```

The combinatorial structure of $D(G, v)$ will generally depend on the choice of v.

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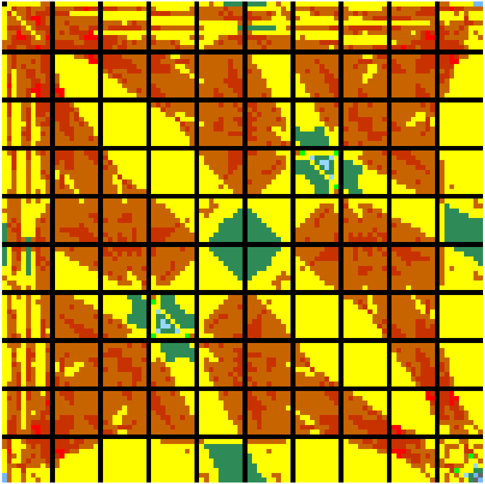
G=SpaceGroup(3,165) has ten possible fundamental domains (depending only on the x and y coordinates of v).



What about discrete groups of isometries with fixed points?

- Crystallographic groups
- Finite subgroups of $O_n(\mathbb{R})$
- $SL_n(\mathcal{O})$ and $PSL_n(\mathcal{O})$ acting on cones of quadratic forms
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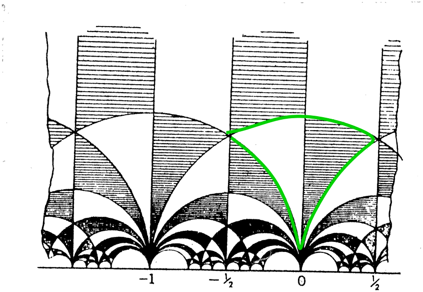
Their cohomology can be computed by adapting an old perturbation lemma of CTC Wall.

Example

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G = SL_2(\mathbb{Z})$ acts on $\mathbb{H} = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$ by

$$z \mapsto \frac{az+b}{cz+d}.$$



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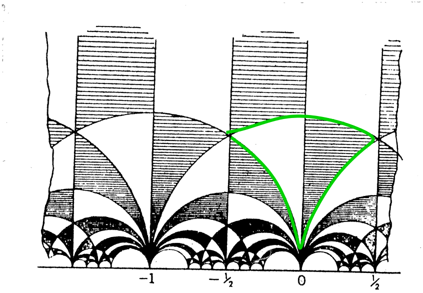
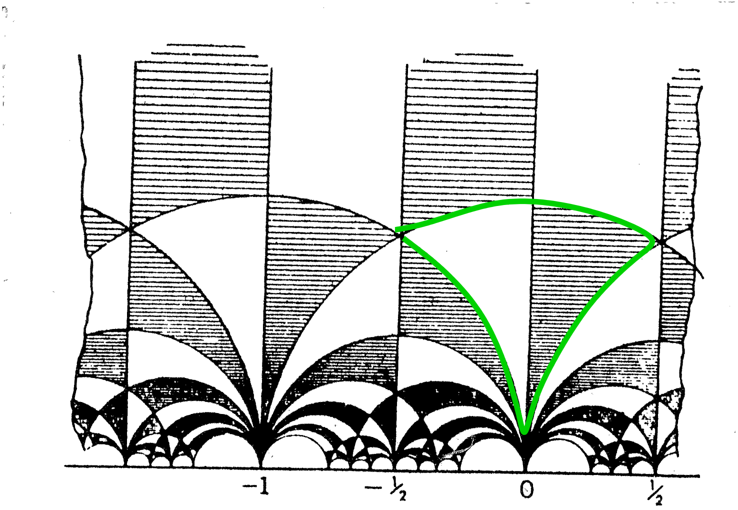


$$\mathbb{H}^* = \mathbb{H} \cup \text{ infinitely many cusps}$$

face stabilizer: C_2 edge stabilizer: C_4 vertex stabilizer: C_6



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$$C_*(\mathbb{H}^*) : 0 \to \mathbb{Z}G \otimes_{\mathbb{Z}C_2} \mathbb{Z} \to \mathbb{Z}G \otimes_{\mathbb{Z}C_4} \mathbb{Z} \to \mathbb{Z}G \otimes_{\mathbb{Z}C_6} \mathbb{Z}$$

Suppose C_2, C_4, C_6 act freely on contractible spaces X, X', X''.

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow \\
C_3(X) & C_3(X') & C_3(X'') \\
\downarrow & \downarrow & \downarrow \\
C_2(X) & C_2(X') & C_2(X'') \\
\downarrow & \downarrow & \downarrow \\
C_1(X) & C_1(X') & C_1(X'') \\
\downarrow & \downarrow & \downarrow \\
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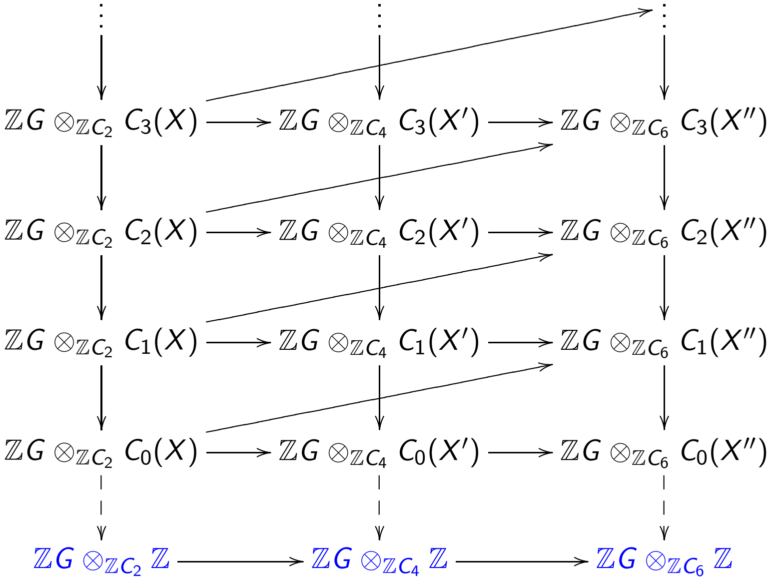

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\end{array}
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Total complex is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$

The resolution

$$R_* = \mathbb{Z}G \otimes_{\mathbb{Z}C_2} C_*(X) \ \oplus \ \mathbb{Z}G \otimes_{\mathbb{Z}C_4} C_{*+1}(X') \ \oplus \ \mathbb{Z}G \otimes_{\mathbb{Z}C_6} C_{*+2}(X'')$$

yields

$$H_{2k+1}(SL_2(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}/12\mathbb{Z}, \quad H_{2k}(SL_2(\mathbb{Z}), \mathbb{Z}) = 0.$$

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$\tilde{\mathbb{H}}^n =$ space of positive definite quadratic forms on $\mathbb{R}^n \subset M_n(\mathbb{R})$.

Action

$$SL_n(\mathbb{R}) \times \tilde{\mathbb{H}}^n \to \tilde{\mathbb{H}}^n, (g, Q) \mapsto gQg^t$$

has finite stabilizer groups.

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$\mathbb{H}^n = \tilde{\mathbb{H}}^n$ modulo scalar multiplication

is a manifold of dimension $n(n+1)/2 - 1$.

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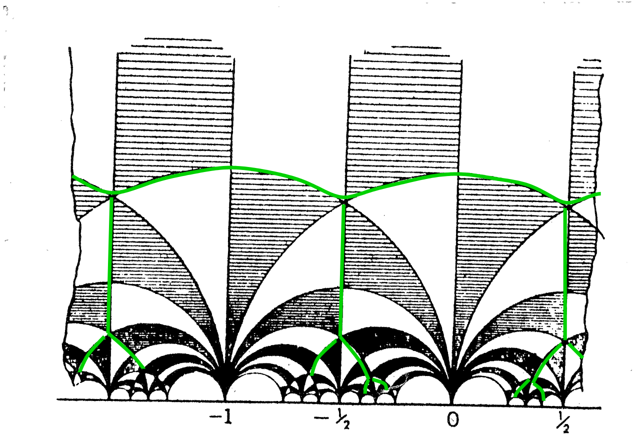

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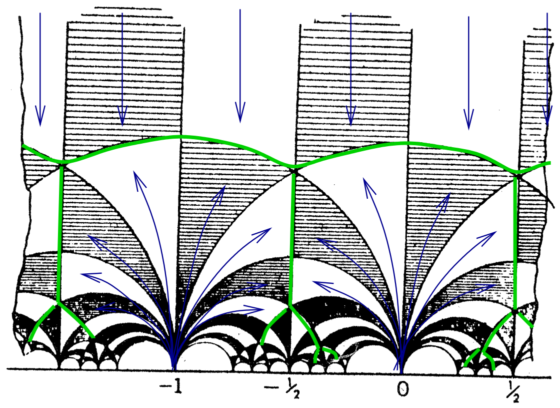

The case $n = 3$

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Work in progress (Dutour, E, Schürmann). For instance

Theorem

$$H_n(PSL(4,\mathbb{Z}),\mathbb{Z}) = \begin{cases} 0 & n = 1 \\ (\mathbb{Z}_2)^3 & n = 2 \\ \mathbb{Z} \oplus (\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_3)^2 \oplus \mathbb{Z}_5, & n = 3 \\ (\mathbb{Z}_2)^4 \oplus \mathbb{Z}_5, & n = 4 \\ (\mathbb{Z}_2)^{13}, & n = 5. \end{cases}$$

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Theorem

$$H_n(PSL(4,\mathbb{Z}),\mathbb{Z})_{(5)} = \begin{cases} \mathbb{Z}_5 & n = 3 + 4k, n = 4 + 4k \\ 0 & \text{otherwise} \end{cases}$$

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gap> Homology(C,5);
[ 2, 2, 2, 2 ]
```
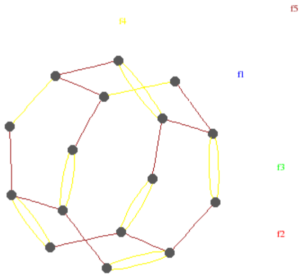

```
gap> PresentationOfResolution(R);
rec( freeGroup := <free group on the generators [ f1, f2, f
  relators := [ f4^2, f4*f1^-2, f3*f1^-1*f3*f1^-1*f3*f1^-1
      f5*f1^-1*f5*f1^-1*f5*f1^-1, f2*f1^-1*f2*f1^-1*f2*f1^-
      f5*f4*f2^-1*f5*f3^-1, f4*f5*f4*f3^-1*f4*f3^-1, f3*f4*
      f4*f5^-1*f4*f3^-1*f4*f5^-1*f4*f3^-1, f3^3, f3*f5*f2,
      f4*f5^-1*f4*f5^-1*f4*f5^-1*f4*f5^-1, f4*f2^2*f5^-1, f
      f4*f2^-1*f4*f2^-1*f4*f2^-1 ] )

gap> IdentityAmongRelatorsDisplay(R,27);
```



Wall's perturbation method: general formulation

Let

$$C_*\colon \to C_n \to C_{n-1} \to \cdots \to C_0$$

be an exact chain complex of $\mathbb{Z}G$-modules C_n with $M = \mathrm{coker}(C_0 \to C_1)$.

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For each p let

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Lemma

There exists a free $\mathbb{Z}G$-resolution $R_* \to M$ with

$$R_n = \bigoplus_{p+q=n} D_{p,q}.$$

A contracting homotopy on R_* is given in terms of contracting homotopies on C_* and on the D_{p*}.

Second Perturbation Application

For a group extension

$$1 \to N \to G \to Q \to 1$$

the data

- $M = \mathbb{Z}$
- $C_* =$ free $\mathbb{Z}Q$ resolution of Z
- $\hat{D}_* =$ free $\mathbb{Z}N$ resolution of Z
- $D_* = \hat{D}_* \otimes_{\mathbb{Z}N} \mathbb{Z}G$

yields a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

Iteration produces a free resolution for nilpotent groups.

Illustration

Theorem

For an odd prime p the group $K_p = \ker(\mathrm{SL}_2(\mathbb{Z}_{p^3}) \to \mathrm{SL}_2(\mathbb{Z}_p))$ has third integral homology group of exponent p^3.

Proof

W. Browder and J. Pakianathan, "Cohomology of uniformly powerful p-groups", (2000).

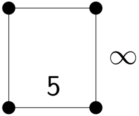
Computer Proof for $p = 5$

```
gap> K5:=MaximalSubgroups(SylowSubgroup(
                                    SL(2,Integers mod 5^3),5))[2];;
gap> Order(K5);
15625

gap> GroupHomology(K5,3);
[ 5, 5, 5, 5, 5, 5, 125 ]
```

Artin groups

Coxeter graphs:



Artin groups

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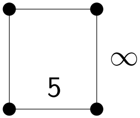
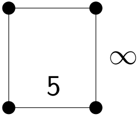


Artin groups:

$$A_D = \langle w, x, y, z \ : \ wxw = xwx, wy = yw, wzw = zwz, xz = zx, yzyzy = zyzyz \rangle$$

Artin groups

Coxeter graphs:



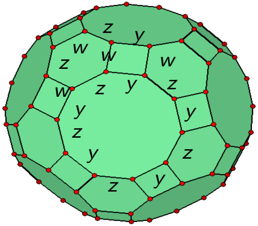
Artin groups:

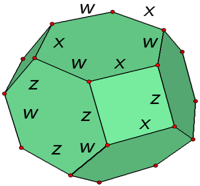
$$A_D = \langle w, x, y, z \;:\; wxw = xwx, wy = yw, wzw = zwz, \\ xz = zx, yzyzy = zyzyz \rangle$$

Coxeter groups:

$$W_D = \langle w, x, y, z \;:\; wxw = xwx, wy = yw, wzw = zwz, \\ xz = zx, yzyzy = zyzyz, \\ w^2 = x^2 = y^2 = z^2 = 1 \rangle$$

Let X_D be the "canonical" path-connected quotient of the two 3-dimensional polytopes:





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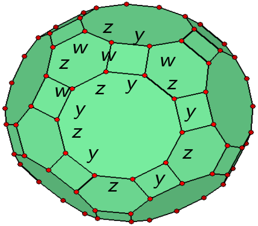

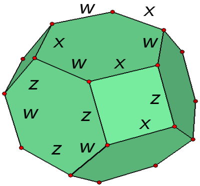

Theorem:

$\pi_1(X_D) \cong A_D$ and $\tilde{X}_D$ is contractible in our example.

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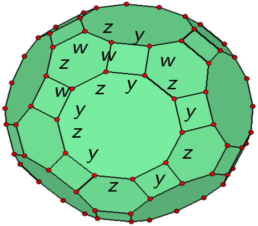

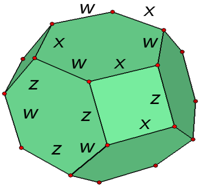

Theorem:
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$K(\pi, 1)$ conjecture (Arnold, Pham, Thom):
$\pi_1(X_D) \cong A_D$ and $\tilde{X}_D$ is contractible in all examples.

van Kampen's theorem: $\pi_1(X_2) \cong A_D$.

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W_D generated by "reflections" in set $\mathcal{A}$ of hyperplanes in $V = \mathbb{E}^n$.

Salvetti (& Davis): $X_D \simeq B_D := \{I \oplus \mathbf{i}V \setminus (\cup_{H \in \mathcal{A}} H \oplus \mathbf{i}H\} \ / \ W_D$

Deligne: $\pi_n(B_D) = 0$ for $n \geq 2$ when W_D is finite.

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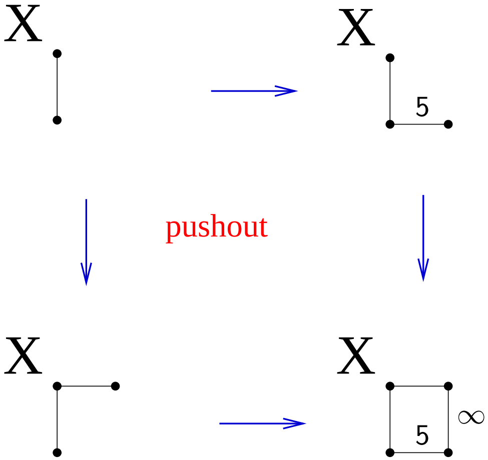
Hendriks, Charney, Davis, Pfeifer, Callegaro, Moroni, Salvetti ...
handle certain infinite W_D.



Whitehead's lemma and results for finite W_D imply $\tilde{X}_D$ contractible in our infinite example.

A subgraph D' of a Coxeter graph D is ∞-free if:

- D' is a connected and full subgraph of D;
- no edge of D' is labelled by ∞.

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Theorem:

An Artin group A_D satisfies the $K(\pi, 1)$ Conjecture if $A_{D'}$ satisfies the conjecture for every ∞-free subgraph D' in D.

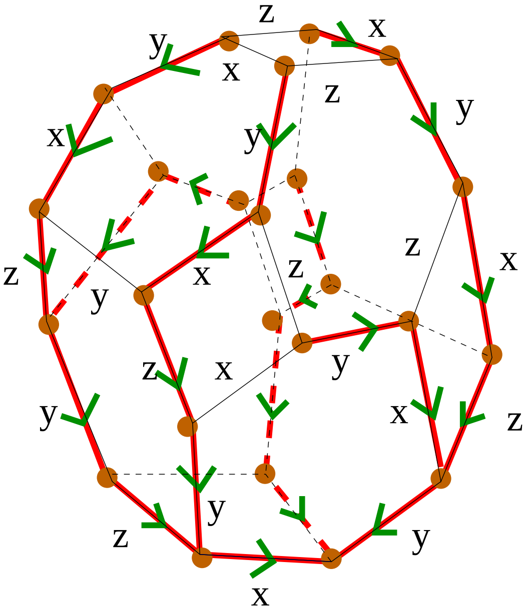
Cohomology of A_D can be computed if it satisfies the conjecture.

Contracting Homotopy on $C_*(\tilde{X}_D)$

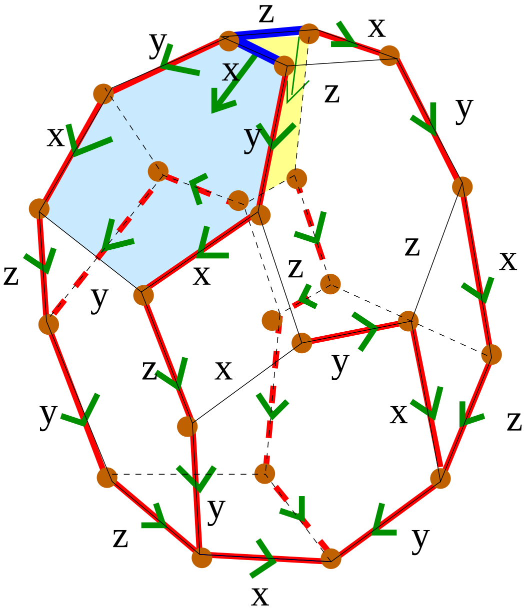
Squier: $H_n(A_D, \mathbb{Z}) \cong H_n(M_D, \mathbb{Z})$ if W_D is finite.

Computations need only handle cells $g \cdot e^n$ with $g \in M_D$.

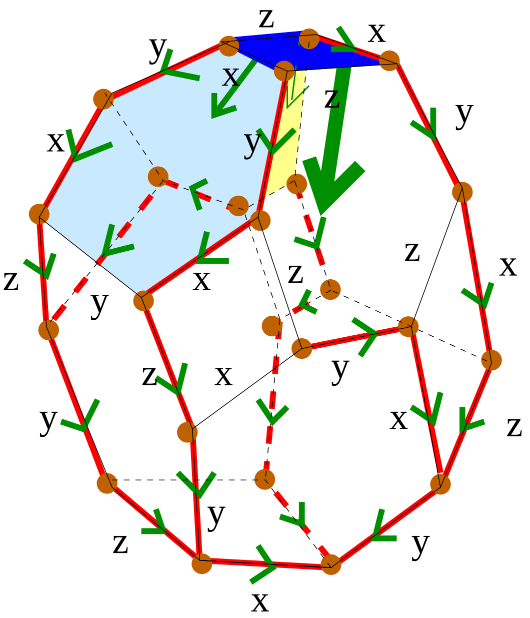
Tree in Cayley graph of M_D starts discrete vector field on $\tilde{X}_D$.



Remaining edges naturally associated to 2-cells in $\tilde{X}_D$.

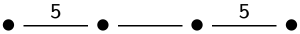


Remaining faces naturally associated to 3-cells in $\tilde{X}_D$.



Illustrative computation

The Artin monoid (group?) G:



has cohomology ring

$$H^*(G, \mathbb{Z}) = \frac{\mathbb{Z}[s, t, u_1, u_2]}{(st + 15u_1 + 15u_2) + J_{\geq 4}}$$

where $|s| = 1$, $|t| = 2$, and $|u_1| = |u_2| = 3$.