Computational Homology II

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Graham Ellis
NUI Galway, Ireland

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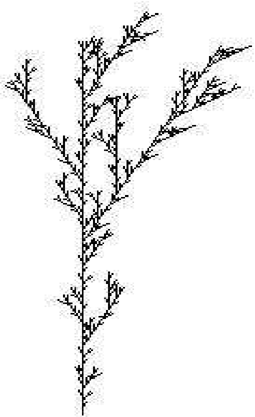
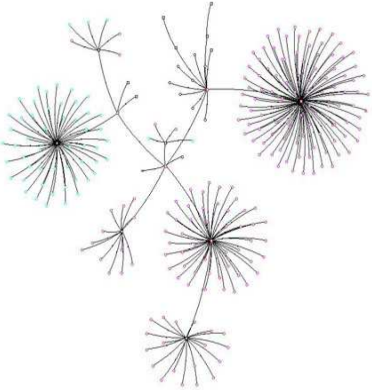
Motivating 2-dimensional example

How do the shapes of the following planar graphs differ?





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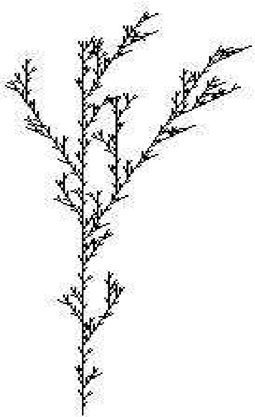
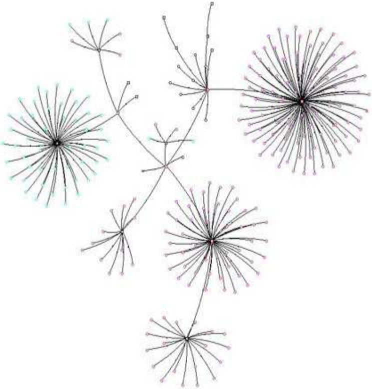
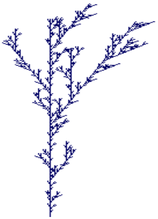
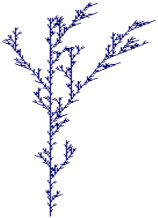
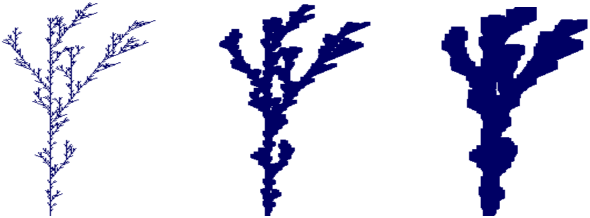


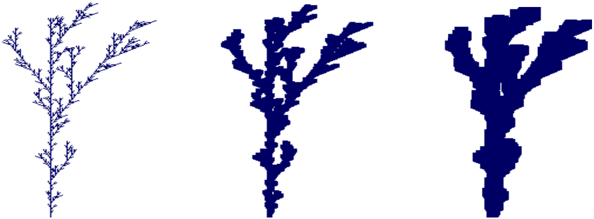


MacPherson: Persistent homology modules can capture shape.

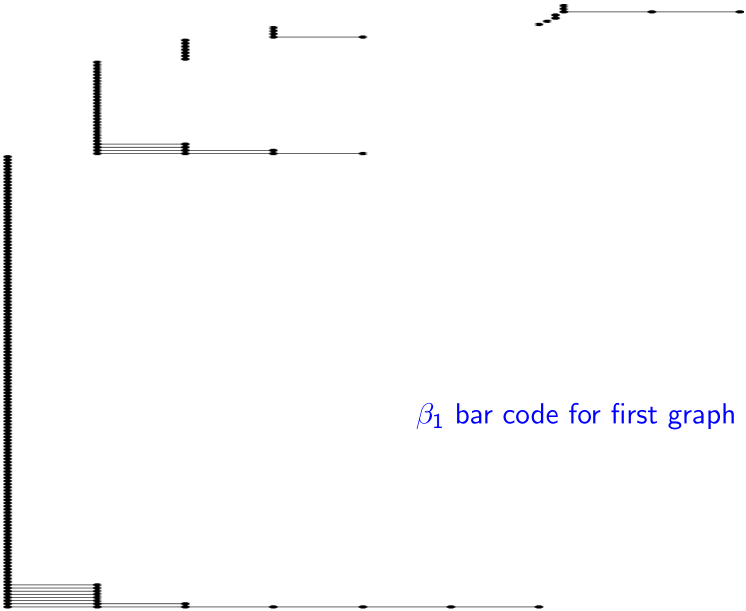


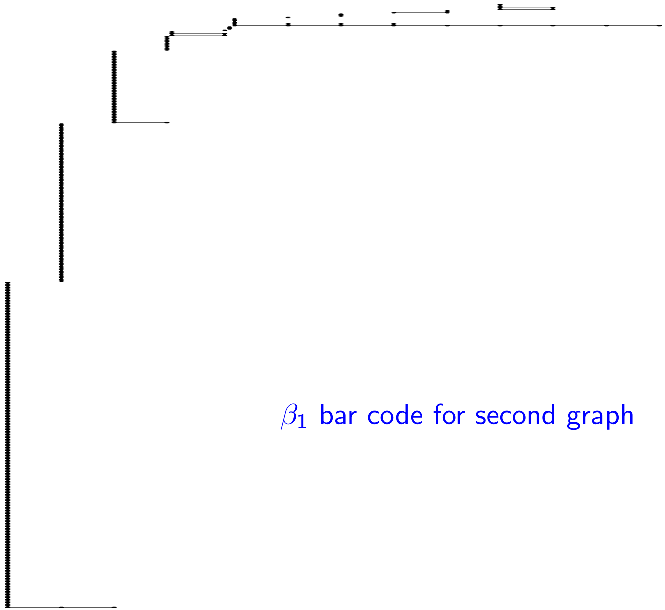




Various thickenings of the first graph

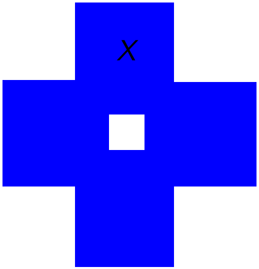


β_1 bar code for first graph



β_1 bar code for second graph

To compute the homlogy of a space



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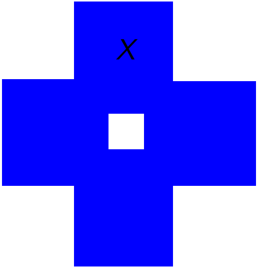

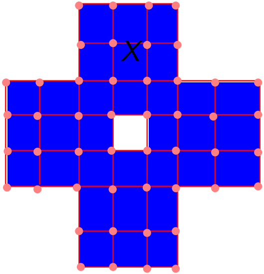

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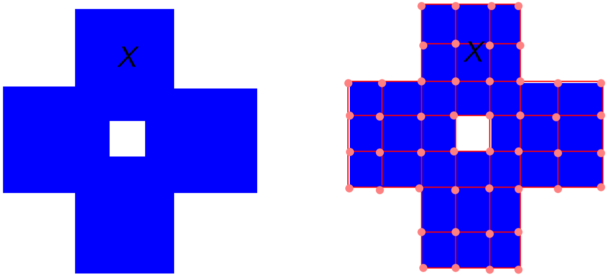

we impose some cell structure, and consider

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$$

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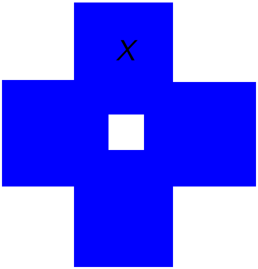

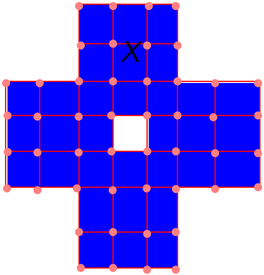

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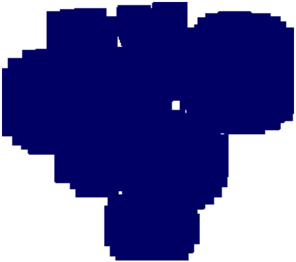
- $H_n(X) = \ker(\partial_n)/\mathrm{image}(\partial_{n+1})$

Our representation of the thickened planar graph $X =$



has 45467 rectangular mesh faces, 91531 edges and 46060 vertices. A naive computation of

$$H_1(X, \mathbb{F}) = \mathbb{F}^5$$

is slow.

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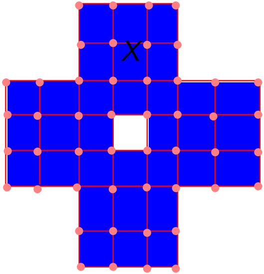
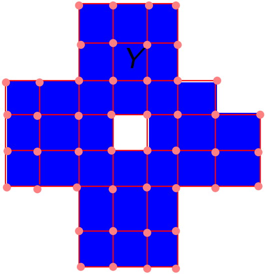
If $X = Y \cup e^n \cup e^{n-1}$ and $Y \cap \overline{e^n} \simeq *$ then $X \simeq Y$.



$\simeq$



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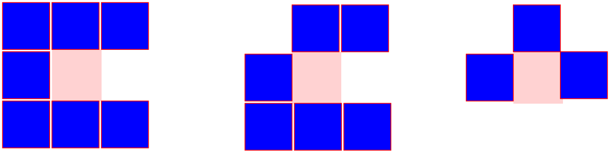
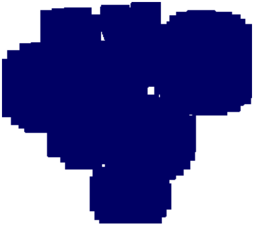
For cubcial $X \subset \mathbb{E}^2$ a cell $\overline{e^2}$ can be deleted without changing homotopy type iff its neighbourhood is one of a storable list:

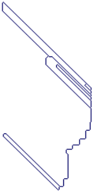


etc.

Our thickened tree has retract



$\simeq$



with 1717 vertices, 2342 edges and 621 faces.

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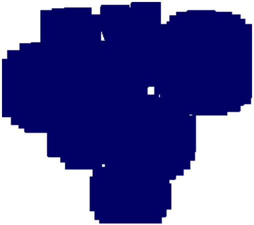

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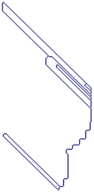

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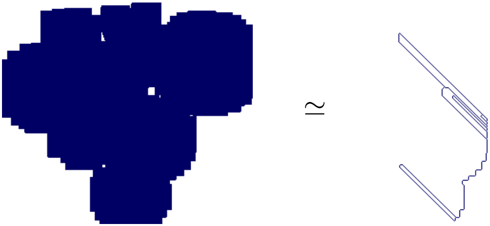

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The computation

$$H_1(X, \mathbb{Z}) \cong H_1(C_*(Y)/C_*(Z)) = \mathbb{Z}^5$$

takes a fraction of a second.

Contracting homotopies

From a homotopy retract $Y \subset X$ we often need

- the chain inclusion $\iota_*\colon C_*(Y) \hookrightarrow C_*(X)$
- its quasi-inverse $\phi_*\colon C_*(X) \to C_*(Y)$
- and a family of homomorphisms

$$h_n\colon C_n(X) \to C_{n+1}(X) \quad (n \geq 0)$$

satisfying

$$\iota_n \phi_n - 1 = \partial_{n+1} h_n + h_{n-1} \partial_n \qquad (h_{-1} = 0).$$

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Forman's Discrete Morse Theory is handy for computing h_n (and hence ϕ_n).

A discrete vector field on a cellular space X is a collection of arrows $s \longrightarrow t$ where

- s, t are cells and any cell is involved in at most one arrow
- $\dim(t) = \dim(s) + 1$
- s lies in the boundary of t

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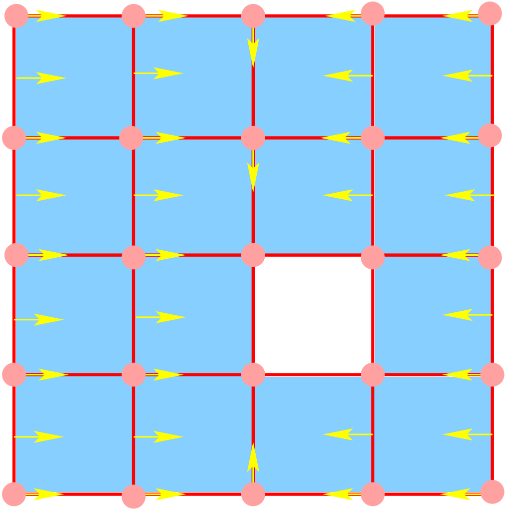

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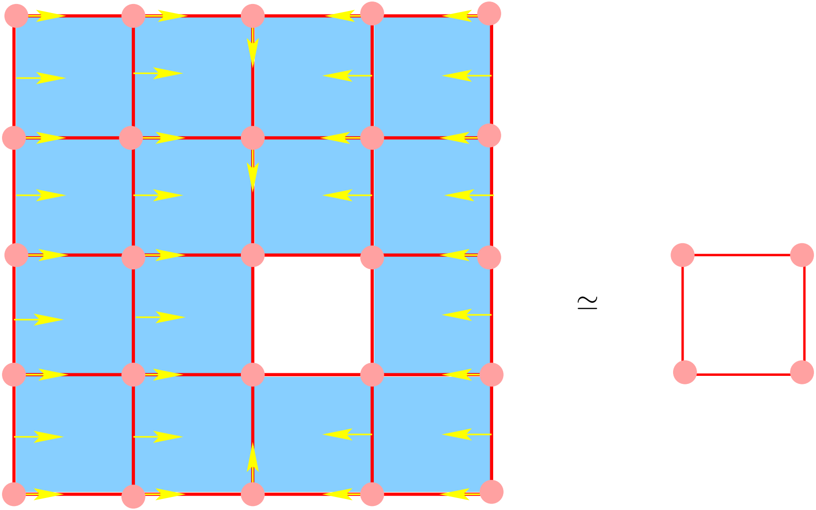
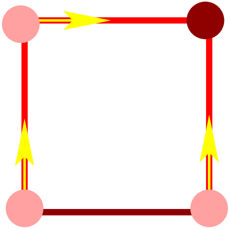


Continued example



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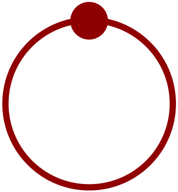

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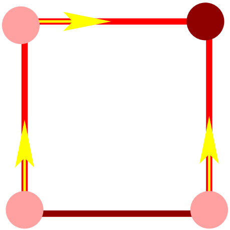
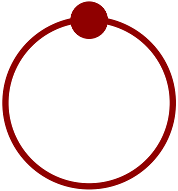


$$\simeq$$



Theorem:

If X is a regular CW-space with discrete vector field then there is a homotopy equivalence

$$X \simeq Y$$

where Y is a CW-space whose cells correspond to those of X not involved in any arrow.

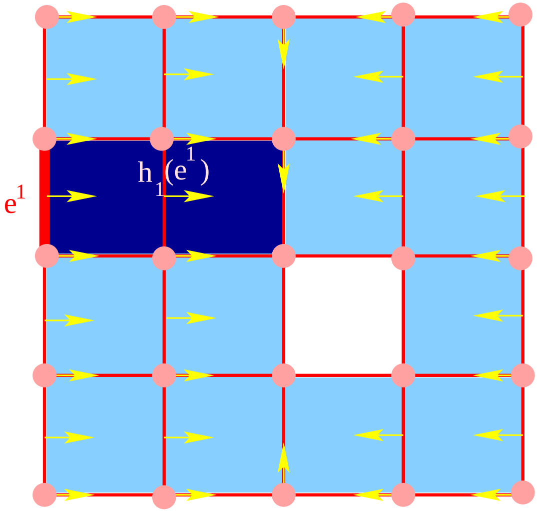
Contracting homotopy

For a discrete vector field arising from a homotopy retraction $Y \subset X$ we define the contracting homotopy

$$h_n \colon C_n(X) \to C_{n+1}(X)$$

on generators e^n by

$$h_n(e^n) = \begin{cases} 0 & \text{if } e^n \text{ is not a source} \\ \sum e_i^{n+1} & \partial_{n+1}(\sum e_i^{n+1}) \text{ contains just one source} \\ & \text{of dimension } n \end{cases}$$



$h_1(e^1)$
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Group (co)homology

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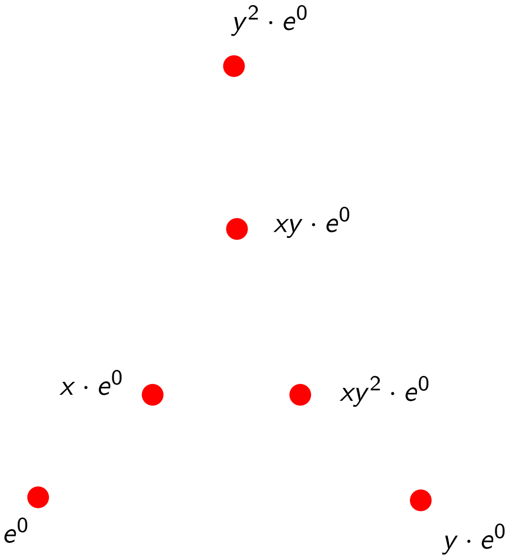
Let's illustrate for $G = S_3$.

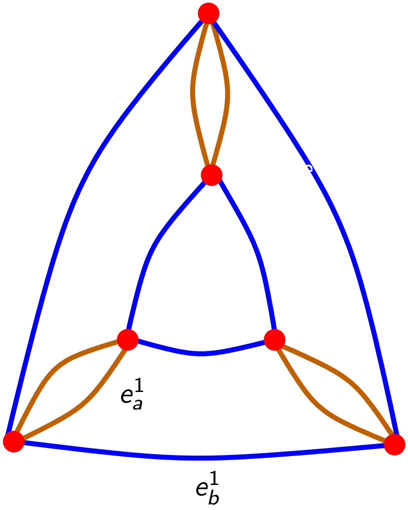
$$x = (1,2), y = (1,2,3)$$

$$G = \langle x, y \rangle$$

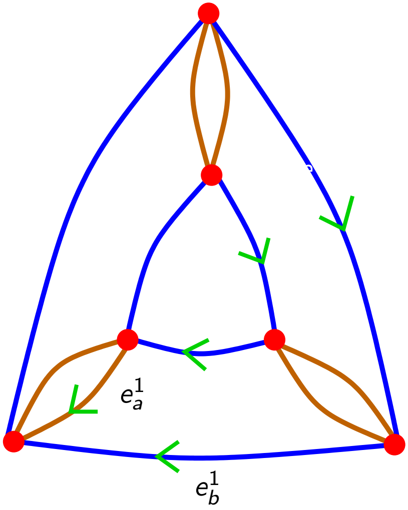
X^0 = one free orbit of vertices



$X^1 = X^0 \cup$ enough free orbits of edges to ensure $\pi_0(X^1) = 0$



Discrete vector field on X^1 ensures $\pi_0(X^1) = 0$.



$X^2 = X^1 \cup$ enough free orbits of 2-cells to ensure $\pi_1(X^2) = 0$

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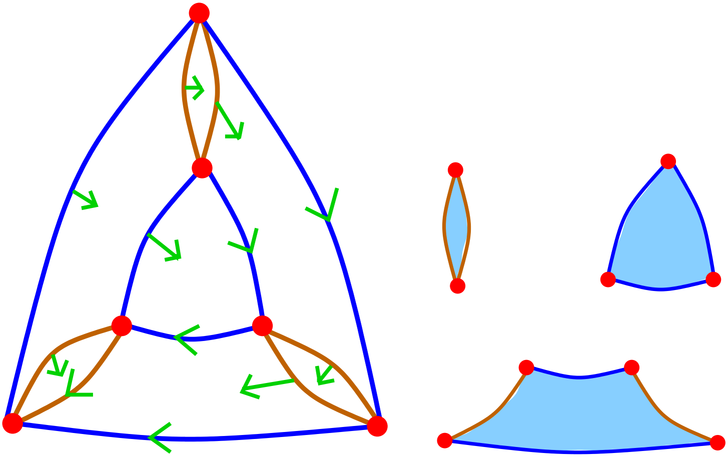
Discrete vector field on X^2



ensures that three orbits suffice.

$X^3 = X^2 \cup$ enough free orbits of 3-cells to ensure $\pi_2(X^3) = 0$

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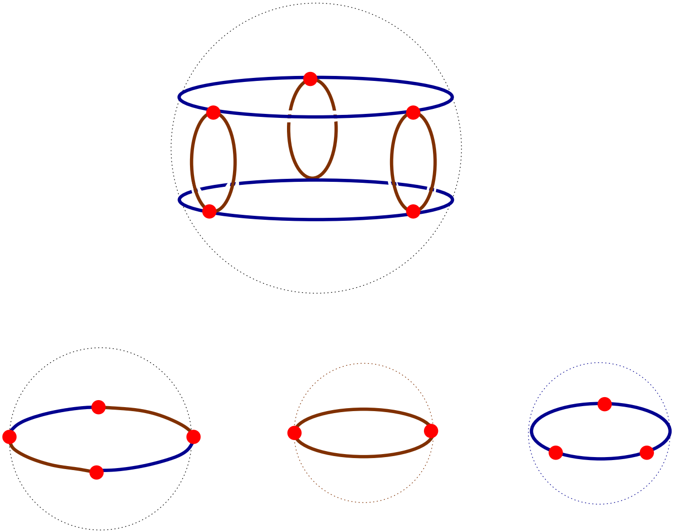
Discrete vector field on X^3 ensures that four orbits suffice.



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is a complex of free $\mathbb{Z}G$-modules with contracting homotopy

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Contracting homotopy needed for induced chain mappings, cup products, ...

A common element of choice:

Let X' be contractible. Choose a homomorphism f_{n+1} so that the following diagram commutes.

$$\begin{array}{ccc} C_{n+1}(X) & \overset{f_{n+1}}{\dashrightarrow} & C_{n+1}(X') \\ \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow \\ C_n(X) & \xrightarrow{f_n} & C_n(X') \end{array}$$

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Choice is algorithmic if some contracting homotopy $h_n\colon C_n(X) \to C_{n+1}(X)$ has already been specified for X'.

$$f_{n+1}(x) = h_n(\ f_n(\ \partial_{n+1}(x)\ )\ )$$

Theorem:

The Mathieu group M_{23} has trivial integral homology $H_n(M_{23}, \mathbb{Z}) = 0$ in dimensions $n = 1, 2, 3, 4$.

Proof:

R.J. Milgram, "The cohomology of the Mathieu group M_{23}", *J. Group Theory* 3 (2000), no. 1, 7–26.

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Computer Proof

```
gap> GroupHomology(MathieuGroup(23),2);
[ ]
gap> GroupHomology(MathieuGroup(23),3);
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gap> GroupHomology(MathieuGroup(23),4);
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gap> GroupHomology(MathieuGroup(23),5);
[ 7 ]
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Analysis of computer proof

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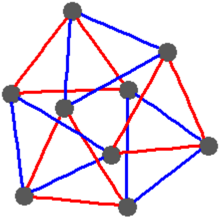
- $X^1_{(3)} =$ 

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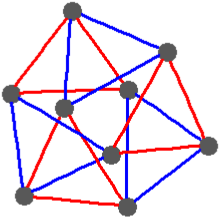
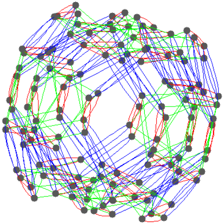
- $X^1_{(3)} =$  $X^1_{(2)} =$ 

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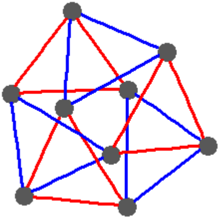
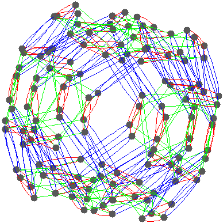
- $X^1_{(3)} =$  $X^1_{(2)} =$ 

- $C_*(X_{(p)})$ is a free $\mathbb{Z}P$-resolution of $\mathbb{Z}$ with contracting homotopy.

- There is a surjection $H_n(P, \mathbb{Z}) \to H_n(G, \mathbb{Z})_{(p)}$ with kernel described (Cartan-Eilenberg) in terms of induced homomorphisms

$$\iota_x \colon H_n(P, \mathbb{Z}) \to H_n(xPx^{-1}, \mathbb{Z})$$

where x ranges over double coset representatives.

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- ι_x constructed using the contracting homotopy.

Conjecture

Any classifying space for an n generator Coxeter group G, whose 2-skeleton corresponds to the standard Coxeter presentation of G, must have at least $\frac{(n+k-1)!}{(n-1)!k!}$ k-dimensional cells.

[M. Salvetti, "Cohomology of Coxeter groups", 2002]

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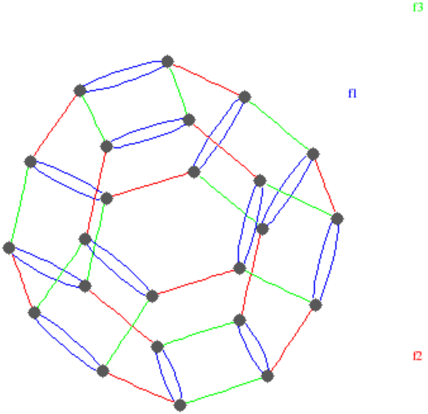
The 3-cells in R are a subset of those in Salvetti's complex. For example:

```
gap> IdentityAmongRelationsDisplay(R,7);
```



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Theorem (King, Green, E): $H^*(Co_3, \mathbb{F}_2)$ has Poincaré series

$$P(t) = \frac{f(t)}{(1-t^8)(1-t^{12})(1-t^{14})(1-t^{15})},$$

where $f(t) \in \mathbb{Z}[t]$ is the monic polynomial of degree 45 with the coefficients 1, 1, 1, 1, 2, 3, 3, 4, 4, 6, 7, 8, 9, 10, 10, 11, 13, 12, 14, 15, 13, 13, 15, 14, 12, 13, 11, 10, 10, 9, 8, 7, 6, 4, 4, 3, 3, 2, 1, 1, 1, 1.

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Proof: Require:

- Free resolution for $Syl_2(Co_3)$
- and completion criteria!

Persistent homology of groups

Group surjections

$$G \twoheadrightarrow G'$$

correspond to classifying space inclusions

$$B(G) = X/G \hookrightarrow B(G) = X'/G' \ .$$

The lower central series

$$L_1(G) = G, \quad L_2(G) = [G, G], \quad \ldots, \quad L_{i+1} = [G, L_i(G)]$$

corresponds to a series of inclusions

$$\cdots \hookrightarrow B(\frac{G}{L_4(G)}) \hookrightarrow B(\frac{G}{L_3(G)}) \hookrightarrow B(\frac{G}{L_2(G)}) \hookrightarrow *$$

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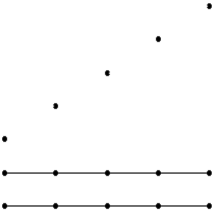
Definition:

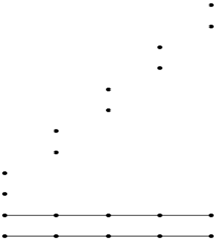
We denote the persistent homology module of these inclusions by

$$H_*^{**}(G, \mathbb{F}) = \{H_n^{ij}(G, \mathbb{F})\}_{n \geq 0, i < j}.$$

$H_1^{**}(D_{32})$

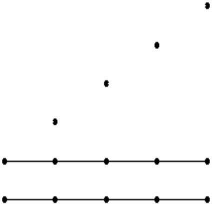


$H_2^{**}(D_{32})$



$H_3^{**}(D_{32})$

$H_1^{**}(Q_{32})$



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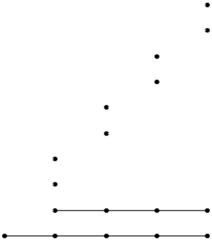

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Proposition:

The invariant $H^{**}_{*}(G, \mathbb{F}_p)$ partitions the 366 prime-power groups of order ≤ 81 into 227 classes with maximum class size equal to 7.

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   $=$ number of bars in β_1 barcode

2. $H_2^{**}(G, \mathbb{F}_p)$ determines rank of $L_c(G)$

   $=$ number of dots in 2nd column of β_2 barcode

3. All β_2 bars start in the first column.

2 & 3 essentially due to Eick and Feichtenschlager.

A group G of order p^n and nilpotency class c has coclass

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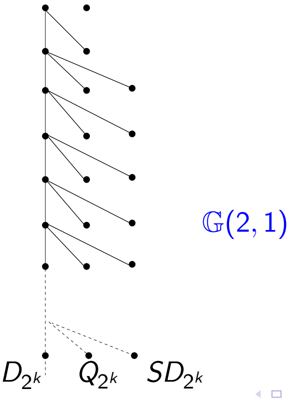
The coclass graph $\mathbb{G}(p, r)$ has as vertices the p-groups of coclass r. Two vertices G, Q are connected by an edge if

$$Q \cong G/L_c(G) \quad \text{with } |L_c(G)| = p \ .$$



Theorem (J. Carlson):

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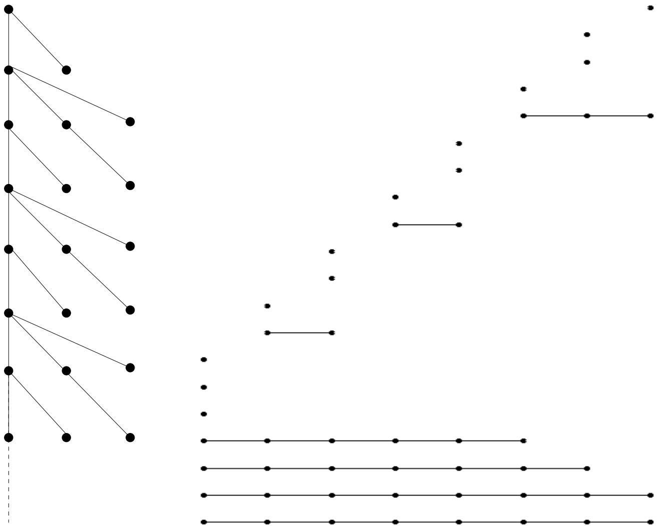
Question:

Does (persistent) homology reflect the structure of coclass trees in a way that would allow us to compute the homology of large p-groups by determining their coclass tree and calculating homology of the initial period of the tree?

A coclass 2 tree and its mainline β_3 bar code



The persistent part of mainline bar codes can be computed from the homology of the p-adic space group arising as the inverse limit of the mainline groups.