

Computational Homology I

Berlin, August 2010

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Step 2 used in applied topology.

One problem from applied topology

Given a set S of points randomly sampled from an unknown manifold M , what can we infer about the topology of M ?

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For instance, $S \subset M \subset \mathbb{E}^2$.



One approach to the problem

Repeatedly “thicken” the set S to produce a sequence of inclusions

$$S = S_1 \subset S_2 \subset S_3 \subset \dots$$

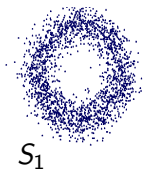
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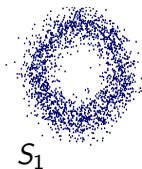


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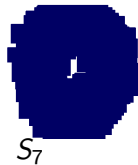
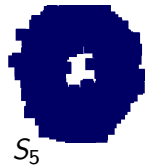
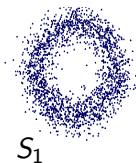


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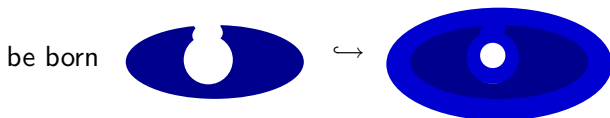
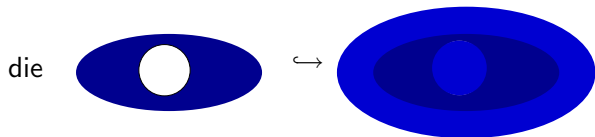
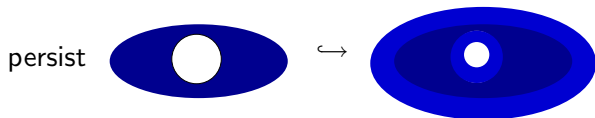
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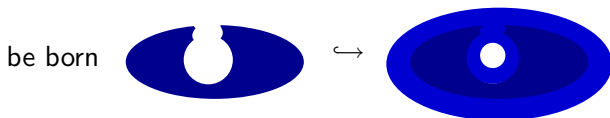
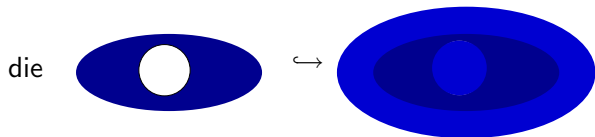
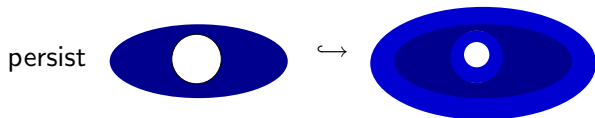
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These numbers are consistent with the sample coming from some region with the homotopy type of a circle.

During an inclusion $S_i \hookrightarrow S_j$ holes can

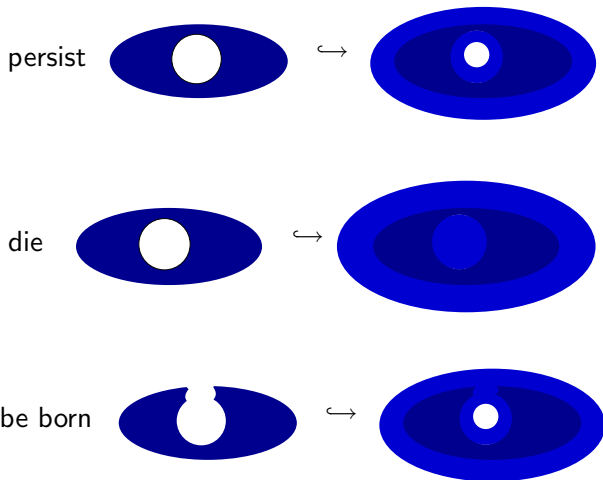


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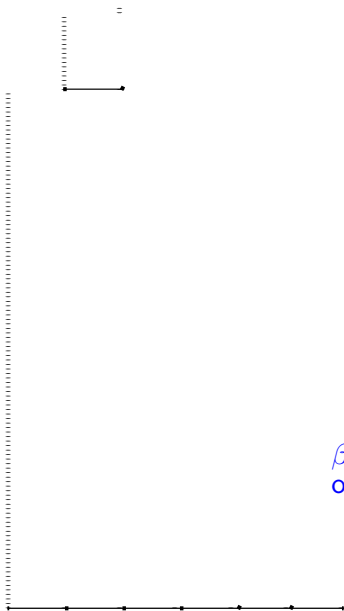
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Bar codes

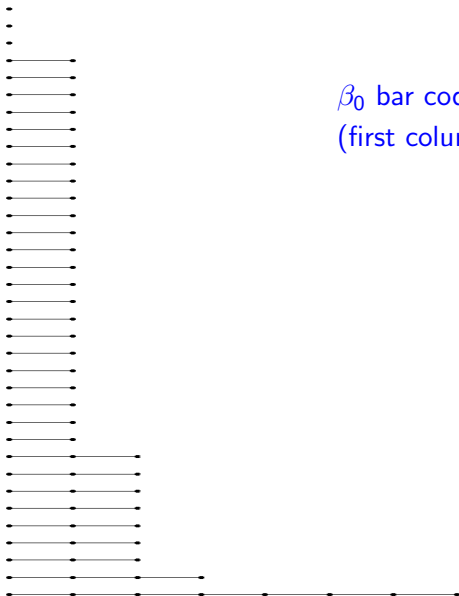
The matrix (β_n^{ij}) can be represented by a graph with horizontal edges and vertices arranged in columns.

The i th column has $\beta_n^{ii} = \beta_n(S_i)$ vertices.

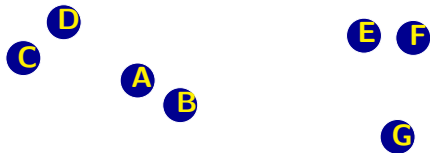
There are β_n^{ij} paths from the i th column to the j th column.



β_1 bar code for
our example



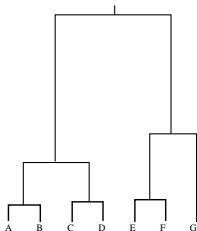
β_0 bar code for our example
(first column cropped)



β_0 bar codes



could be enhanced to dendrograms (or phylogenetic trees)

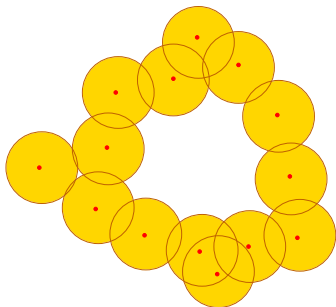


From data to spaces

Given a set $S \subset \mathbb{E}^n$ and number $\epsilon > 0$ consider

$$X_\epsilon^S = \bigcup_{x \in S} B(x, \epsilon)$$

with $B(x, \epsilon)$ the ball of radius ϵ centred on x .

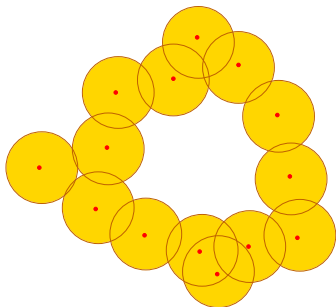


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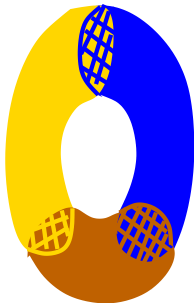
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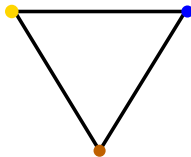
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Any $0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \dots$ yields inclusions $X_{\epsilon_1}^S \subset X_{\epsilon_2}^S \subset X_{\epsilon_3}^S \subset \dots$

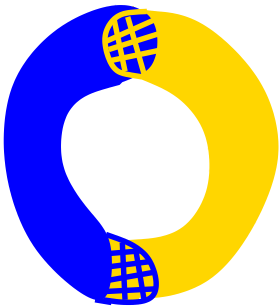


\cong



vertices \leftrightarrow coloured regions

edges \leftrightarrow intersections



\neq

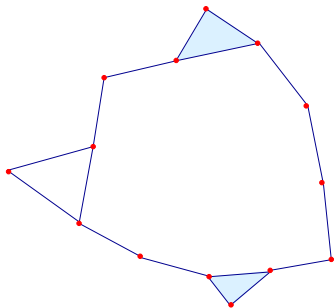
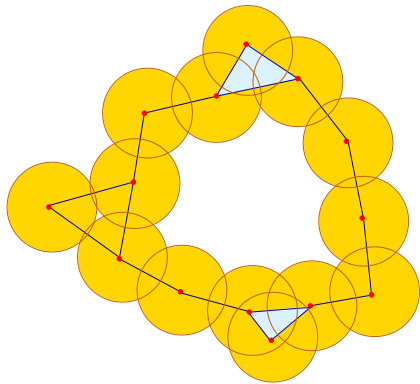


edges \leftrightarrow (non-contractible) intersections

Cech complex NX_ϵ^S

NX_ϵ^S is the simplicial complex with vertex set S and one simplex for each $\sigma \subseteq S$ satisfying

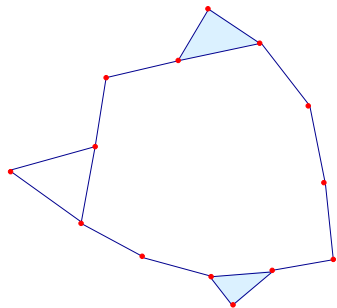
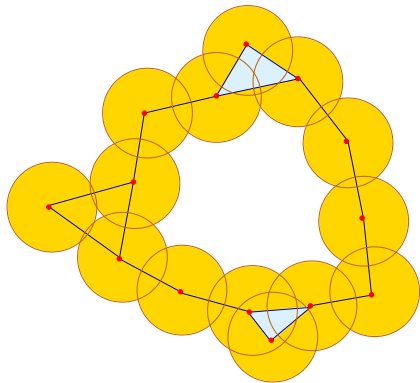
$$\bigcap_{x \in \sigma} B(x, \epsilon) \neq \emptyset.$$



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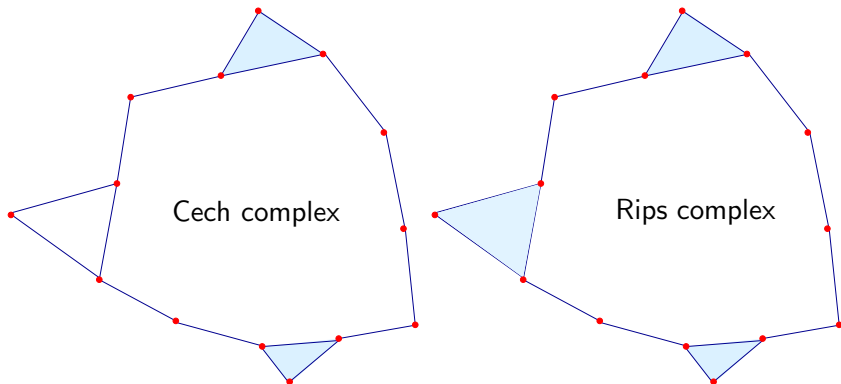


Theorem: $X_\epsilon^S \cong NX_\epsilon^S$

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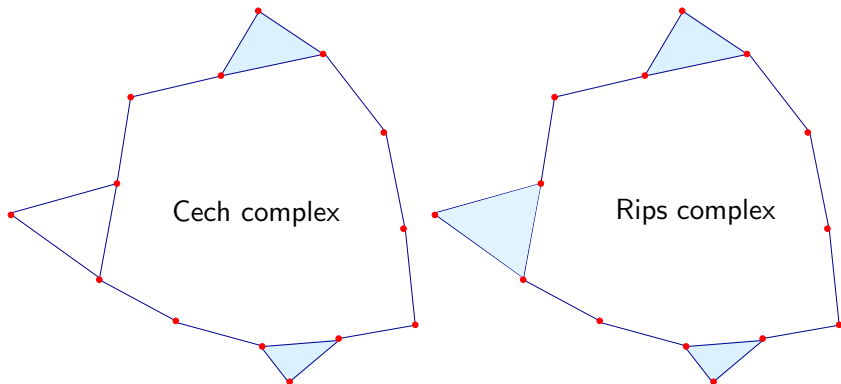
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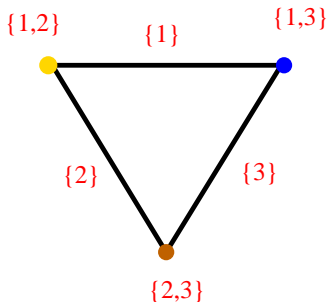
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Theorem: $R_\epsilon^S \subseteq NX_{\epsilon\sqrt{2}}^S \subseteq R_{\epsilon\sqrt{2}}^S$

A *simplicial complex* K is a collection of subsets of some (ordered) set S satisfying:

- ▶ $\{x\} \in K$ for all $x \in S$,
- ▶ if $\sigma' \subset \sigma \in K$ then $\sigma' \in K$.



Computing Betti numbers

The chain complex $C_*(K)$

$$\longrightarrow C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \longrightarrow \dots \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

▶ $C_n(K)$ = vector space with basis $\{\sigma \in K : |\sigma| = n + 1\}$

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$$\partial_n(\{x_1, \dots, x_{n+1}\}) = \sum_{i=1}^{n+1} (-1)^i \{x_1, \dots, \hat{x}_i, \dots, x_{n+1}\}$$

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and *Betti numbers*

$$\beta_n(K) = \dim(H_n(K)).$$

Filtrations

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_N$$

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$$\beta_n^{ij} = \dim(H_n^{ij}(K)).$$

Theorem: All β_n^{ij} can be determined from semi-echelon forms of the two matrices ∂_n^N and ∂_{n+1}^N .

Example

$$\partial_n^4 = \left(\begin{array}{cccc|ccc|cc|ccc} \times & \times & \times & \times & \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & & \times & \times & \times \\ \hline & & & & \times & \times & \times & & \times & \times & \times \\ & & & & \times & \times & \times & & \times & \times & \times \\ & & & & \times & \times & \times & & \times & \times & \times \\ \hline & & & & & & & & \times & \times & \times \\ & & & & & & & & \times & \times & \times \\ & & & & & & & & \times & \times & \times \\ \hline & & & & & & & & & \times & \times & \times \\ & & & & & & & & & \times & \times & \times \end{array} \right)$$

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x & x & x & x & x & x & x & & x & x & x \\
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& & & & x & x & x & & x & x & x \\
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& & & & & & & & & x & x & x \\
& & & & & & & & & x & x & x
\end{array} \right)$$

$$\dim(C_n(K_4)) = 12, \quad \dim(C_{n-1}(K_4)) = 14, \quad \dim(C_n(K_3)) = 9, \quad \dots$$

Column reduce

$$\partial_n^4 = \begin{pmatrix} \begin{array}{cc|ccc} x & x & & & \\ & x & & & \\ & & x & & \\ & & & x & \\ & & & & x \end{array} & \begin{array}{ccc|ccc} x & x & x & & & \\ x & x & x & & & \\ x & x & x & & & \\ x & x & x & & & \\ x & x & x & & & \\ x & x & x & & & \end{array} & \begin{array}{cc|ccc} x & x & & & \\ & x & & & \\ & & x & & \\ & & & x & \\ & & & & x \end{array} & \begin{array}{ccc|ccc} x & x & x & & & \\ x & x & x & & & \\ x & x & x & & & \\ x & x & x & & & \\ x & x & x & & & \\ x & x & x & & & \end{array} \end{pmatrix}$$

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$$\dim(\ker(\partial_n^4)) = 3, \quad \dim(\text{image}(\partial_n^4) \cap C_{n-1}(K_2)) = 7, \quad \dots$$

Illustration: Natural Images

van Hateren & van der Schaaf: 4167 digital photos of random outdoor scenes.

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Mumford, Lee, Pedersen: 8 000 000 random high-contrast 3×3 patches in \mathbb{E}^9 .

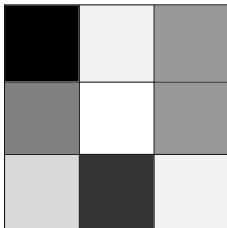
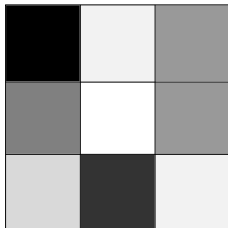


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By normalizing with respect to brightness and contrast the patches are projected onto a set of points \mathcal{M} in a topological seven-sphere $S^7 \subset \mathbb{E}^8$.

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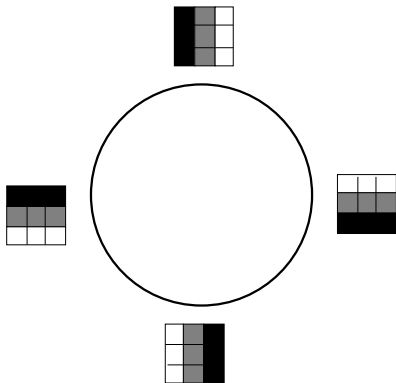
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$\mathcal{M}(k)$ contains the 25% of patches with least $\delta_k(x)$.

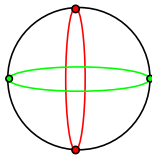
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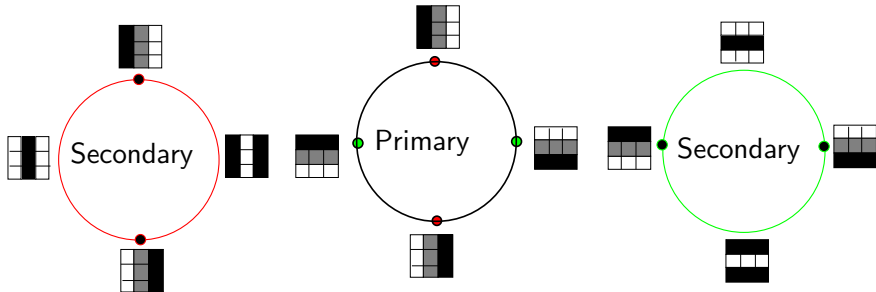
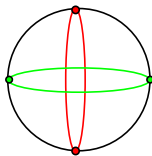


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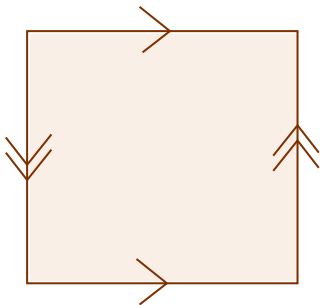
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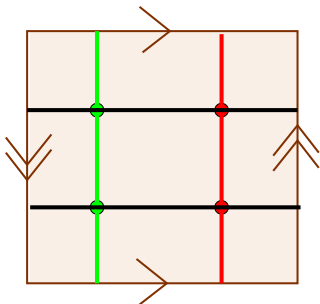
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$$H_2(\text{Klein bottle}, \mathbb{Z}_2) = \mathbb{Z}_2$$



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$$V \cong \bigoplus_i t^{a_i} \cdot \mathbb{F}[t] \oplus \left(\bigoplus_j t^{b_j} \cdot (\mathbb{F}[t]/(t^{c_j} \cdot \mathbb{F}[t])) \right).$$

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Remark: $\mathbb{Z}[t]$ and $\mathbb{F}[s, t]$ are not PIDs.

Quivers

A persistence module V is a representation of the quiver

$$1 \longrightarrow 2 \longrightarrow \dots \longrightarrow N$$

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Finite type

A quiver has *finite type* if it has only finitely many isomorphism types of indecomposable representations.

Gabriel's theorem

A connected quiver is of finite type if and only if its underlying graph is of type A_n , D_n , E_6 , E_7 , E_8 .

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Use bar codes for small sub samples S_1, S_2, \dots of a large sample $S \subset M$ to infer about M .

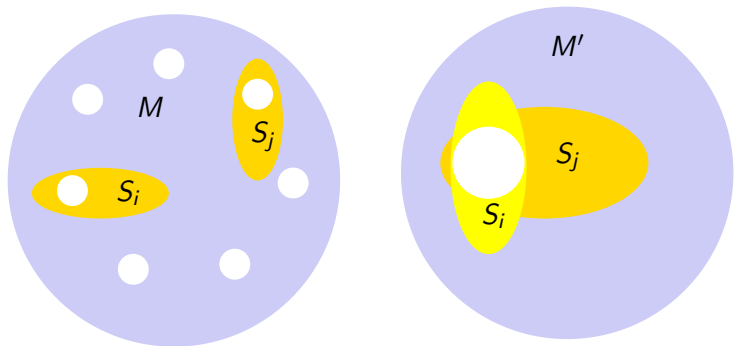
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Difficulty: If bar codes for S_i suggest $\beta_1 = 1$ the situation could be M or M' .



Carlsson, de Silva: Consider inclusions

$$S_1 \rightarrow S_1 \cup S_2 \leftarrow S_2 \rightarrow S_2 \cup S_3 \leftarrow \dots \leftarrow S_N$$

and persistence in the *ziz-zag diagram*

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Gabriel's theorem: The quiver

$$1 \rightarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow \dots \leftarrow N$$

has indecomposable representations $\mathbb{I}(i, j)$ for $1 \leq i < j \leq N$:

$$\mathbb{I}(1, 2) : \mathbb{F} \xrightarrow{1} \mathbb{F} \leftarrow 0 \rightarrow 0 \leftarrow \dots \leftarrow 0$$

$$\mathbb{I}(1, 3) : \mathbb{F} \xrightarrow{1} \mathbb{F} \xleftarrow{1} \mathbb{F} \rightarrow 0 \leftarrow \dots \leftarrow 0$$

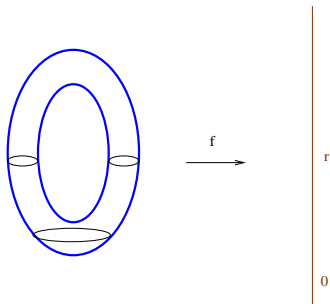
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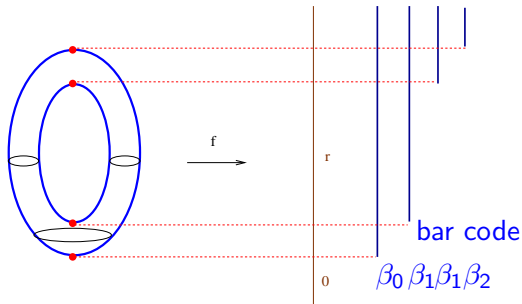
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