Discrete Vector Fields & Classifying Spaces

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NUI Galway

Definition: A *classifying space* of a discrete group G is a quotient

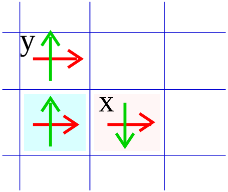
$$BG = EG/\sim$$

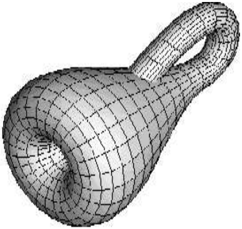
of any contractible space EG by a free action of G.

Definition: $$H^n(G, \mathbb{Z}) = H^n(BG, \mathbb{Z})$$

Example 1

$$G = \langle x, y | yxy = x \rangle$$





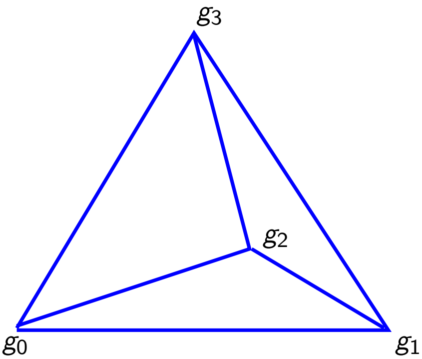
$EG = \mathbb{R}^2$

BG = Klein bottle

$$H_n(G, \mathbb{Z}) = H_n(\text{Klein bottle}, \mathbb{Z})$$

Example 2

For any G could use a simplicial space EG with one k-simplex for each $k+1$-tuple $(g_0, \cdots g_k) \in G \times \cdots \times G$.



Example 3 (Bui, E)

$$H_{100}(\ SL_2(\mathbb{Z}[1/7])\ ,\mathbb{Z}) = \mathbb{Z}_3$$

```
gap> EG:=ResolutionSL2Z(7,101);;
gap> BG:=TensorWithIntegers(EG);
gap> Homology(BG,100);
[ 3 ]
```

Example 4 (E, King)

$$\sum_{n\geq 0} a_n\, x^n = \frac{1}{-x^3+3x^2-3x+1}$$

$$\text{where} \quad a_n = \dim\ H^n(Syl_2(M_{12}), \mathbb{Z}_2)$$

```
gap> G:=SylowSubgroup(MathieuGroup(12),2);;
gap> PoincareSeries(G);
(1)/(-x^3+3*x^2-3*x+1)
```

Q1: How do we compute homology of BG ?

Q2: How do we compute EG ?

Q3: How do we compute induced maps $BG \longrightarrow BG'$?

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A: Discrete vector fields

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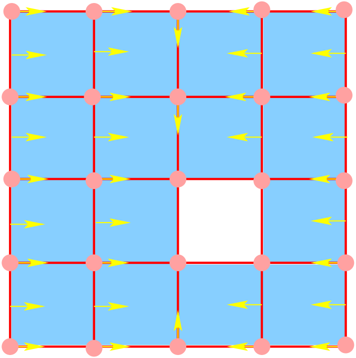
Q3: How do we compute induced maps $BG \longrightarrow BG'$?
A: Discrete vector fields

A discrete vector field on a regular CW-space X is a collection of arrows $s \to t$ where

- s, t are cells and any cell is involved in at most one arrow
- $\dim(t) = \dim(s) + 1$ and s lies in the boundary of t



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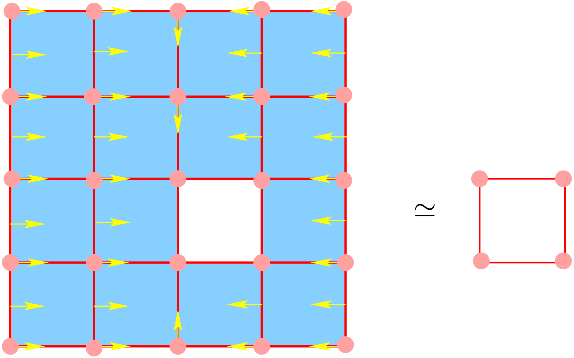

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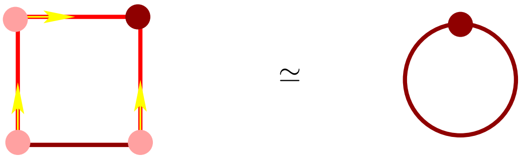
Continued example



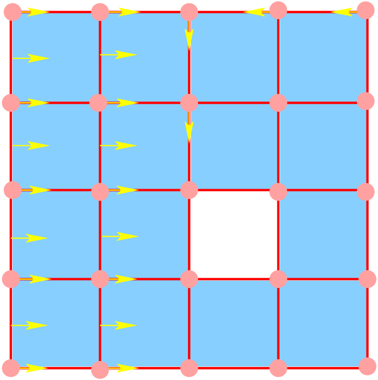
Theorem: (Whitehead/Forman)
If X is a regular CW-space with admissible discrete vector field then there is a homotopy equivalence

$$X \simeq Y$$

where Y is a CW-space whose cells correspond to those of X not involved in any arrow.

Recursive construction of an admissible vector field

Input a regular CW-space X with (partial) vector field.



The critical cells are those not involved in arrows.

Recursive construction of an admissible vector field

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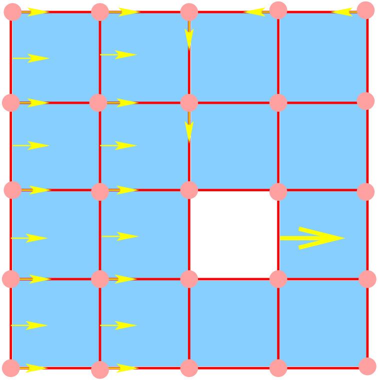
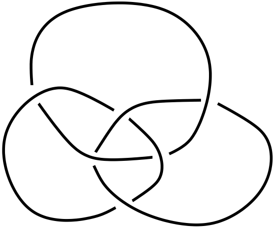


The critical cells are those not involved in arrows.

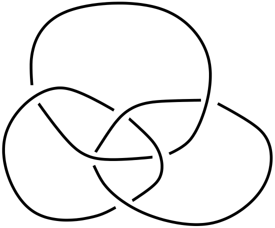
Choose a critical cell s which is in the boundary of exactly one higher dimensional critical cell t and add the arrow $s \rightarrow t$.

Illustration of an implementation:



```
gap> L:=ReadLinkAsPureCubicalComplex("image.jpg");;
gap> C:=ComplementOfPureCubicalComplex(L);;
gap> Y:=CubicalComplexToRegularCWSpace(C);;
gap> Size(Y);
12419107
gap> CriticalCellsOfRegularCWSpace(Y);
[ [ 2, 334 ], [ 2, 115000 ], [ 2, 139630 ],
[ 1, 386713 ], [ 1, 404957 ], [ 1, 405056 ],
[ 1, 600331 ], [ 0, 164802 ], [ 0, 241782 ] ]
```

Illustration of an implementation:



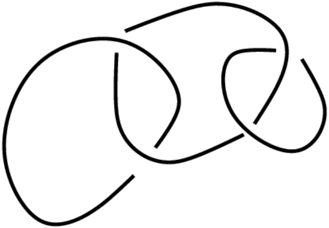
```
gap> F:=FundamentalGroup(Y);;

gap> RelatorsOfFpGroup(F);
[ f3^-1*f2^-1*f3*f1^-1*f2*f3*f2^-1*f3^-1*f1*f2,
  f2^-1*f3^-1*f1*f3*f1^-1*f2*f1*f3^-1*f1^-1*f3 ]
```

$$\pi_1 Y = \langle x, y, z \mid x^{-1}[y, z]x = [y, z^{-1}],\ y^{-1}[z^{-1}, x]y = [x, z^{-1}] \rangle$$

$$\text{where } [x, y] = xyx^{-1}y^{-1}$$

Illustration of an implementation:



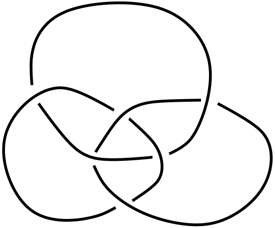
```
gap> F:=FundamentalGroup(Y);;

gap> RelatorsOfFpGroup(F);
[ f2^-1*f1*f2*f1^-1, f3^-1*f1*f2^-1*f3*f1^-1*f2 ]
```

$$\pi_1 Y = \langle x, y, z \mid [y^{-1}, x] = 1,\ z^{-1}(xy^{-1})z = y^{-1}x \ \rangle$$

$$\text{where } [x, y] = xyx^{-1}y^{-1}$$

Illustration of an implementation:



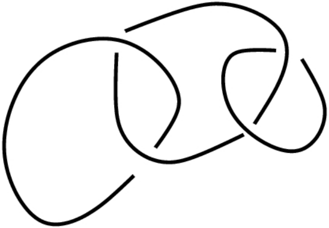
```
gap> F:=FundamentalGroup(Y);;

gap> GroupHomology(NilpotentQuotient(F,3),3);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
  0, 0, 0, 0, 0, 0]
```

$$H_3(\ \pi_1 Y/[[\pi_1 Y, \pi_1 Y], \pi_1 Y]\ , \mathbb{Z}) = \mathbb{Z}^{40}$$

where $Y = \mathbb{R}^3 - \mathrm{Borromean\ rings}$

Illustration of an implementation:



```
gap> F:=FundamentalGroup(Y);;

gap> GroupHomology(NilpotentQuotient(F,3),3);
[ 0, 0, 0, 0, 0, 0 ]
```

$$H_3(\ \pi_1 Y/[[\pi_1 Y, \pi_1 Y], \pi_1 Y]\ \ , \mathbb{Z}) = \mathbb{Z}^6$$

where $Y = \mathbb{R}^3 - \text{chain of three links}$

Algorithm for computing EG for finite/automatic G

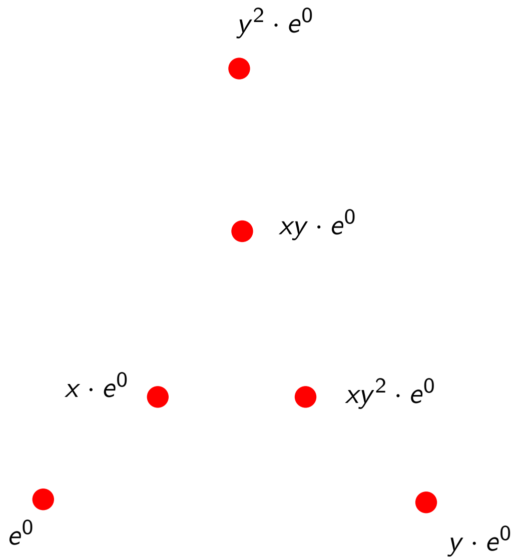
Useful theorem: X is contractible iff $\pi_n(X^{n+1}) = 0$ for $n \geq 0$.

Algorithm: Recursively construct $X^0 \subset X^1 \subset X^2 \subset \cdots \subset X = EG$ using discrete vector fields.

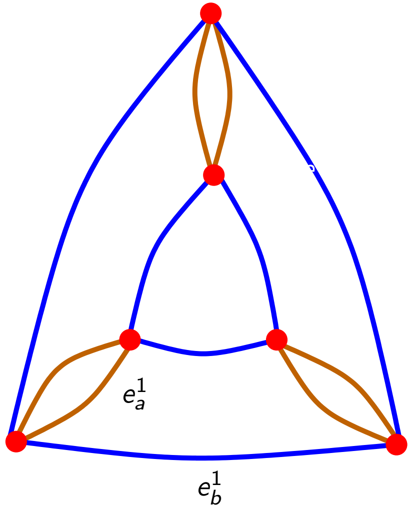
Let's illustrate algorithm on $G = S_3$ with generators

$$x = (1, 2), y = (1, 2, 3).$$

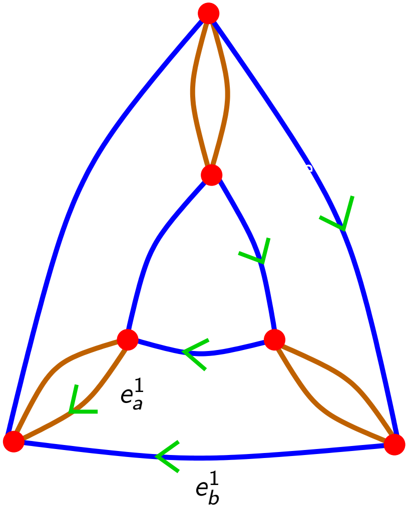
X^0 = one free orbit of vertices



$X^1 = X^0 \cup$ enough free orbits of edges to ensure $\pi_0(X^1) = 0$

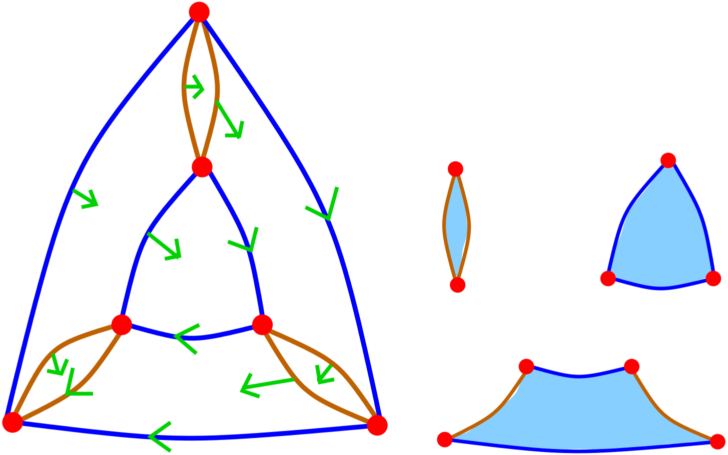


Discrete vector field on X^1 ensures $\pi_0(X^1) = 0$.



$X^2 = X^1 \cup$ enough free orbits of 2-cells to ensure $\pi_1(X^2) = 0$

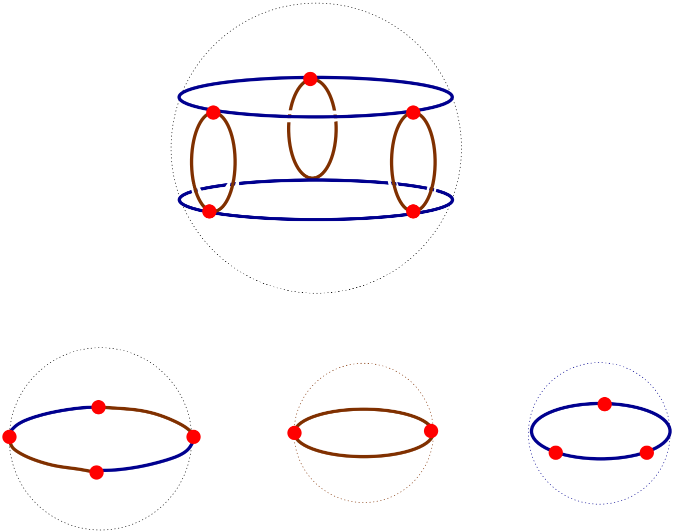
Discrete vector field on X^2



ensures that three orbits suffice.

$X^3 = X^2 \cup$ enough free orbits of 3-cells to ensure $\pi_2(X^3) = 0$

Discrete vector field on X^3 ensures that four orbits suffice.



Algorithm yields $X = EG$ and

$$C_*(X): \quad \cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to 0$$

is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$.

The discrete vector field on X is represented as a contracting homotopy

$$h_n\colon C_n(X) \to C_{n+1}(X) \quad (n \geq 0)$$

$$h_n\partial_{n+1} + \partial_{n+1}h_n = 1$$

Algorithm for induced maps $EG \longrightarrow EG'$

Element of choice

For each $x \in \ker(d_n \colon C_n(X) \to C_{n-1}(X))$ *choose an element* $\tilde{x} \in C_{n+1}(X)$ *such that*

$$d_{n+1}(\tilde{x}) = x.$$

handled by

Setting $\tilde{x} = h_n(x)$ ensures $d_{n+1}(\tilde{x}) = x$.

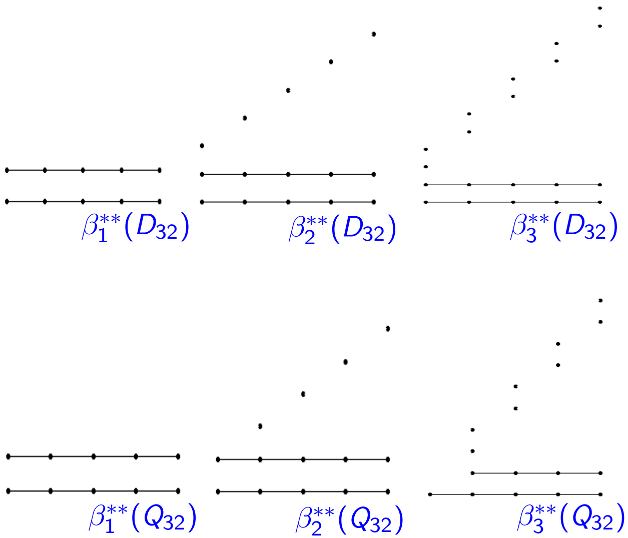
Illustration

$$\gamma_1 G = G,$$
$$\gamma_{i+1} G = [G, \gamma_i G] = \langle\ gxg^{-1}x^{-1}\ :\ g \in G, x \in \gamma_i G\ \rangle$$

$$\cdots \ \rightarrow\ B(G/\gamma_4 G)\ \rightarrow\ B(G/\gamma_3 G)\ \rightarrow\ B(G/\gamma_2 G)\ \rightarrow\ B(1)$$

Definition: The *persistent Betti numbers* are

$$\beta_n^{ij} G = rank(\ \ H_n(G/\gamma_i G, \mathbb{F}) \rightarrow H_n(G/\gamma_j G, \mathbb{F})\ \ )$$



$\beta_1^{**}(D_{32})$
$\beta_2^{**}(D_{32})$
$\beta_3^{**}(D_{32})$
$\beta_1^{**}(Q_{32})$
$\beta_2^{**}(Q_{32})$
$\beta_3^{**}(Q_{32})$

PROPOSITION:

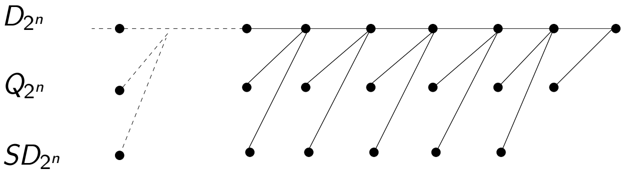
The invariants β_*^{**} over $\mathbb{F}_p$ partition the 366 prime-power groups of order ≤ 81 into 227 classes with maximum class size equal to 7.

On these groups the invariants $\{\beta_n^{**}\}_{6\geq n\geq 1}$ are as strong as the invariants $\{\beta_n^{**}\}_{n\geq 1}$.

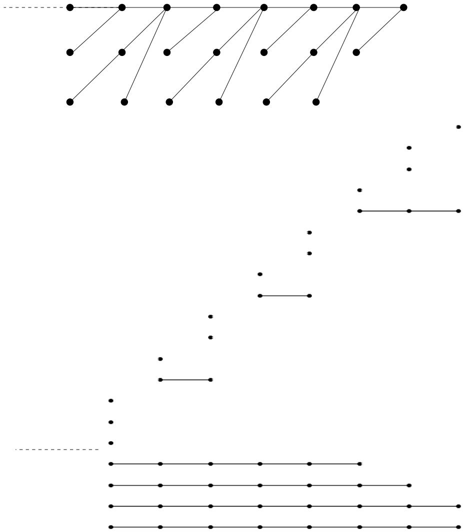
A coclass tree is a path component in the graph $\mathbb{G}(p)$ with

$$\text{Vertex set} = \text{all finite } p\text{-groups}$$

$$\text{edge from } G \text{ to } Q \text{ if } |\gamma_{class(G)}(G)| = p \text{ and } Q \cong G/\gamma_{class(G)}(G)$$



Relationship between persistent homology and coclass?

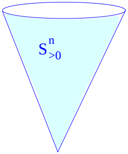


A coclass tree and β_3 bar code for its mainline groups

Algorithm for constructing EG when $G = SL_n(\mathbb{Z})$

(Dutour-Sikrić, E & Schürmann)

A real $n \times n$ symmetric matrix Q is positive definite if $v^t Q v > 0$ for all $v \in \mathbb{R}^2 \setminus \{0\}$. Let $S^n_{>0}$ denote the space of all such matrices.



$Q \in S^n_{>0}$, $v \in \mathbb{Z}^n$:	*EXAMPLE* $Q = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$
$Q[v] = v^t Q v$	$Q[x, y] = x^2 + xy + y^2$
$\min(Q) = \min_{0 \neq v \in \mathbb{Z}^n} Q[v]$	$\min(Q) = 1$
$\mathrm{Min}(Q) = \{v \in \mathbb{Z}^n \ : \ Q[v] = \min(Q)\}$	six vectors
$\rho(v) = v^t v \in S^n_{>0}$	$\rho(1, 1) : x^2 + 2xy + y^2$

$Q \in S^n_{>0}$ is perfect if

$$P[v] = \min(Q) \text{ for all } v \in \mathrm{Min}(Q)$$

implies $P = Q$.

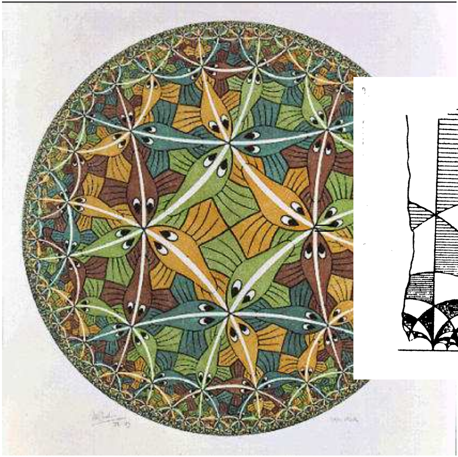
THEOREM (Voronoi)
There are only finitely many perfect forms Q up to $SL_n(\mathbb{Z})$-equivalence, and the cells

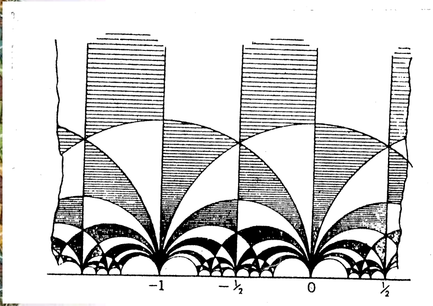
$$\mathrm{Dom}(Q) = \left\{ \sum_{v \in \mathrm{Min}(Q)} \lambda_v \rho(v) \ : \ \lambda_v \geq 0 \right\}$$

tessellate (the rational closure of) $S^n_{>0}$.

$S^n_{=1} = \{A \in S^n_{>0} \ : \ \min(A) = 1\}$ is a piecewise linear surface of dimension $\binom{n+1}{2} - 1$.

For $S^2_{=1}$ we have:





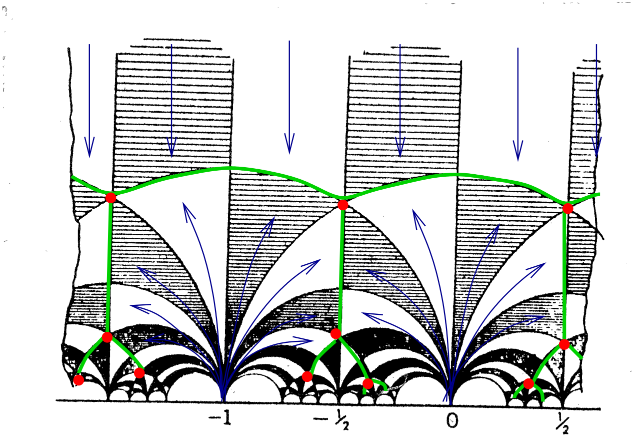
$P \in SL_n(\mathbb{Z})$ acts on $A \in S^n_{=1}$ by

$$A \mapsto PAP^t$$

$A \in S^n_{=1}$ is well rounded if there are linearly independent $v_1, ..., v_n$ with $A[v_i] = 1$.

Avner Ash: there is an $SL_n(\mathbb{Z})$-invariant $\binom{n}{2}$-dimensional homotopy retract $S^n_{wr} \subset S^n_{=1}$

S^2_{wr} =



$G = SL_n(\mathbb{Z})$ acts cellularly on the contractible CW-complex S^n_{wr} each cell e having finite stabilizer group G_e.

We can compute contractible EG_e with contracting vector field.

THEOREM

The space S^n_{wr} and spaces EG_e (one for each orbit representative e) can be algorithmically combined into a contractible CW-space EG with fixed point free cellular action of $SL_n(\mathbb{Z})$. If S^n_{wr} is given a contracting vector field then EG inherits one too.

PROPOSITION

$$H_5(PSL_4(\mathbb{Z}), \mathbb{Z}) = (\mathbb{Z}_2)^{13}$$

Background to the theorem

A *homotopy equivalence data*

$$(C,d) \underset{i}{\overset{p}{\leftrightarrows}} (B,d),\ h \qquad (*)$$

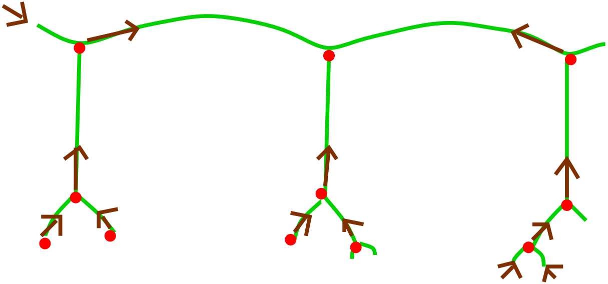
consists of chain complexes C, B, quasi-isomorphisms i, p and a homotopy $ip - 1 = dh + hd$. A *perturbation* on $(*)$ is a homomorphism $\epsilon\colon B \to B$ of degree -1 such that $(d+\epsilon)^2 = 0$.

PERTURBATION LEMMA: (M. Crainic) If $M = (1-\epsilon h)^{-1}\epsilon$ exists then

$$(C,d') \underset{i'}{\overset{p'}{\leftrightarrows}} (B,d+\epsilon),\ h' \qquad (**)$$

is a homotopy equivalence data where

$$i' = i + hMi,\quad p' = p + pMh,\quad h' = h + hMh,\quad d' = d + pMi\quad .$$



Contracting homotopy on S^2_{wr} implemented as an algorithm for expressing an element of $SL_2(\mathbb{Z})$ in terms of generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Example of DVF (K. Conrad) To express

$$A = \begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix}$$

in terms of

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

note that $17 = 2 \cdot 7 + 3$ and so

$$T^{-2}A = \begin{pmatrix} 3 & 5 \\ 7 & 12 \end{pmatrix}$$

$$ST^{-2}A = \begin{pmatrix} -7 & -12 \\ 3 & 5 \end{pmatrix}$$

$$\cdots$$

$$ST^2ST^2ST^3ST^{-2}A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} = -T$$

Algorithm for constructing EG when $G = SL_2(\mathbb{Z}[1/m])$

(Bui, E)

Serre: For integer $m > 0$ and prime $gcd(p, m) = 1$

$$SL_2(\mathbb{Z}[1/pm]) \cong SL_2(\mathbb{Z}[1/m]) *_{\Gamma_0(p)} SL_2(\mathbb{Z}[1/m]).$$

where

$$\Gamma_0 = \left\{ A \in SL_2(\mathbb{Z}[1/m]) \ : \ A \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \ mod \ p \right\}$$

has finite index in $SL_2(\mathbb{Z}[1/m])$.

LEMMA: A total space EG with contracting vector field can be built (algorithmically) for an amalgamated free product

$$G = A *_\Gamma B$$

from total spaces EA, EB and $E\Gamma$ and a contracting vector field on the tree associated to the amalgamated product.

Remarks:

We have $ESL_2(\mathbb{Z})$.

Hence we have (an inefficient) $E\Gamma_0(p) = ESL_2(\mathbb{Z})$.

Discrete vector fields should simplify the cell structure on $E\Gamma_0(p)$.

We can thus construct $ESL_2(\mathbb{Z}[1/p])$ and then $ESL_2(\mathbb{Z}[1/m])$.

However:

We'd like to compute $H_3(SL_2(\mathbb{Z}[1/2.3.5.7.11]),\mathbb{Z})$ to get a feel for $H_3(SL_2(\mathbb{Q}),\mathbb{Z})$.

```
gap> EG:=ResolutionSL2Z(2*3*5*7*11,4);
Resolution of length 4 in characteristic 0
for SL(2,Z[1/2310]) .

gap> List([0..4],EG!.dimension);
[ 64, 1152, 8240, 30720, 66816 ]
```

THANK YOU